

# A TWISTED FIRST HOMOLOGY GROUP OF THE HANDLEBODY MAPPING CLASS GROUP

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**ABSTRACT.** Let  $H_g$  be a 3-dimensional handlebody of genus  $g$ . We determine the twisted first homology group of the mapping class group of  $H_g$  with coefficients in the first integral homology group of the boundary surface  $\partial H_g$  for  $g \geq 2$ .

## 1. INTRODUCTION

Let  $H_g$  be a 3-dimensional handlebody of genus  $g$ , and  $\Sigma_g$  the boundary surface  $\partial H_g$ . We denote by  $\mathcal{H}_g$  and  $\mathcal{M}_g$  the mapping class group of  $H_g$  and the boundary surface  $\Sigma_g$ , respectively. These are the groups of isotopy classes of orientation preserving homeomorphisms of  $\Sigma_g$  and  $H_g$ . Let  $D$  be a closed 2-disk in the boundary  $\Sigma_g$  of the handlebody, and pick a point  $*$  in  $\text{Int } D$ . Let us denote by  $\mathcal{H}_g^*$  and  $\mathcal{H}_{g,1}$  the groups of the isotopy classes of orientation preserving homeomorphisms of  $H_g$  fixing  $*$  and  $D$  pointwise, respectively. We also denote by  $\mathcal{M}_g^*$  and  $\mathcal{M}_{g,1}$  the groups of the isotopy classes of orientation preserving homeomorphisms of  $\Sigma_g$  fixing  $*$  and  $D$  pointwise, respectively. We use integral coefficients for homology groups unless specified throughout the paper.

In the cases of the mapping class group  $\mathcal{M}_g^*$  and  $\mathcal{M}_g$  of a surface  $\Sigma_g$ , Morita [12, Corollary 5.4] determined the first homology group with coefficients in the first integral homology group of the surface. Morita [13, Remark 6.3] extended the first Johnson homomorphism to a crossed homomorphism  $\mathcal{M}_g^* \rightarrow \frac{1}{2}\Lambda^3(H_1(\Sigma_g))$ , and showed that the contraction of this crossed homomorphism gives isomorphisms  $H_1(\mathcal{M}_g^*; H_1(\Sigma_g)) \cong \mathbb{Z}$  and  $H_1(\mathcal{M}_g; H_1(\Sigma_g)) \cong \mathbb{Z}/(2g-2)\mathbb{Z}$  when  $g \geq 2$ . For twisted homology groups of the mapping class groups of nonorientable surfaces, see Stukow [18]. In the cases of the automorphism group  $\text{Aut } F_n$  and the outer automorphism group  $\text{Out } F_n$  of a free group  $F_n$  of rank  $n$ , Satoh [17] computed  $H_1(\text{Aut } F_n; H^1(F_n))$  and  $H_1(\text{Out } F_n; H^1(F_n))$  for  $n \geq 2$ . Kawazumi [8] extended the first Andreaskis-Johnson homomorphism to a crossed homomorphism  $\text{Aut } F_n \rightarrow H^1(F_n) \otimes H_1(F_n)^{\otimes 2}$ . The contraction of this crossed homomorphism also gives isomorphisms  $H_1(\text{Aut } F_n; H^1(F_n)) \cong \mathbb{Z}$  and  $H_1(\text{Out } F_n; H^1(F_n)) \cong \mathbb{Z}/(n-1)\mathbb{Z}$ .

In this paper, we compute the twisted first homology groups of  $\mathcal{H}_g$  and  $\mathcal{H}_g^*$  with coefficients in the first integral homology group of the boundary surface  $\Sigma_g$ . Note that the restrictions

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of homeomorphisms of  $H_g$  to  $\Sigma_g$  induce an injective homomorphism  $\mathcal{H}_g \rightarrow \mathcal{M}_g$ , and we treat the group  $\mathcal{H}_g$  as a subgroup of  $\mathcal{M}_g$ . The followings are main theorems in this paper.

**Theorem 1.1.**

$$H_1(\mathcal{H}_g; H_1(\Sigma_g)) \cong \begin{cases} \mathbb{Z}/(2g-2)\mathbb{Z} & \text{if } g \geq 4, \\ \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } g = 3, \\ (\mathbb{Z}/2\mathbb{Z})^2 & \text{if } g = 2, \end{cases}$$

Furthermore, when  $g \geq 4$ , the homomorphism  $H_1(\mathcal{H}_g; H_1(\Sigma_g)) \rightarrow H_1(\mathcal{M}_g; H_1(\Sigma_g))$  induced by the inclusion is an isomorphism. When  $g = 2, 3$ , this homomorphism is surjective and the kernel is isomorphic to  $\mathbb{Z}/2\mathbb{Z}$ .

**Theorem 1.2.**

$$H_1(\mathcal{H}_{g,1}; H_1(\Sigma_g)) \cong H_1(\mathcal{H}_g^*; H_1(\Sigma_g)) \cong \begin{cases} \mathbb{Z} & \text{if } g \geq 4, \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } g = 2, 3. \end{cases}$$

Furthermore, when  $g \geq 4$ , the homomorphism  $H_1(\mathcal{H}_g^*; H_1(\Sigma_g)) \rightarrow H_1(\mathcal{M}_g^*; H_1(\Sigma_g))$  induced by the inclusion is an isomorphism. When  $g = 2, 3$ , this homomorphism is surjective and the kernel is isomorphic to  $\mathbb{Z}/2\mathbb{Z}$ .

In this paper, we also study relationships between the second homology groups of  $\mathcal{H}_g$ ,  $\mathcal{H}_g^*$ , and  $\mathcal{H}_{g,1}$ . The second homology group of  $\mathcal{M}_g$  is calculated by Harer [3] when  $g \geq 5$ . It contains some minor mistakes and these are corrected in [4] later. For surfaces with an arbitrary number of punctures and boundary components, see Korkmaz-Stipsicz [9]. See also Benson-Cohen [1] and Sakasai [16] for low genera. There are some results which imply that the cohomology group of the handlebody mapping class group  $\mathcal{H}_g$  is similar to that of  $\mathcal{M}_g$ . Morita [11, Proposition 3.1] showed that the rational cohomology group of any subgroup of the mapping class group decomposes into a direct sum. Later, Kawazumi-Morita [7, Proposition 5.2] generalized it to the cohomology group with coefficients in  $A = \mathbb{Z}[1/(2g-2)]$ . In particular, the cohomology group of the handlebody mapping class group with a puncture decomposes as

$$H^n(\mathcal{H}_g^*; A) \cong H^n(\mathcal{H}_g; A) \oplus H^{n-1}(\mathcal{H}_g; H^1(\Sigma_g; A)) \oplus H^{n-2}(\mathcal{H}_g; A).$$

Hatcher-Wahl [5] showed that the integral cohomology groups of the mapping class groups of 3-manifolds stabilize in more general settings. Hatcher also announced that the rational stable cohomology group coincides with the polynomial ring generated by the even Morita-Mumford classes. However, as far as we know, even the second integral homology group of handlebody mapping class groups has not been computed yet.

Here is the outline of our paper:

In Section 2, we investigate the relationship between the second integral homology group of the handlebody mapping class group fixing a point or a 2-disk in  $\Sigma_g$  pointwise with that of  $\mathcal{H}_g$  using Theorem 1.1.

In Section 3, we compute the twisted first homology group  $H_1(\mathcal{H}_g; H_1(\Sigma_g))$  to prove Theorem 1.1 in the case when  $g \geq 4$ . We also compute the twisted first homology groups of  $\mathcal{H}_g$  with coefficients in  $\text{Ker}(H_1(\Sigma_g) \rightarrow H_1(H_g))$  and  $H_1(H_g)$ .

Let  $\mathcal{L}_g$  denote the kernel of the homomorphism  $\mathcal{H}_g \rightarrow \text{Out}(\pi_1 H_g)$ . The exact sequence

$$1 \longrightarrow \mathcal{L}_g \longrightarrow \mathcal{H}_g \longrightarrow \text{Out}(\pi_1 H_g) \longrightarrow 1$$

induces exact sequences between their first homology groups with coefficients in  $\text{Ker}(H_1(\Sigma_g) \rightarrow H_1(H_g))$  and  $H_1(H_g)$ . Luft [10] showed that the group  $\mathcal{L}_g$  coincides with the twist group, which is generated by Dehn twists along meridian disks. Satoh [17] determined the twisted first homology groups  $H_1(\text{Out } F_n; H_1(F_n))$  and  $H_1(\text{Out } F_n; H^1(F_n))$ . Applying Luft's and Satoh's results to the exact sequences, we can determine  $H_1(\mathcal{H}_g; H_1(\Sigma_g))$  when  $g \geq 4$ .

In Section 4, we review a finite presentation of the handlebody mapping class group  $\mathcal{H}_g$  given by Wajnryb [19].

In Section 5, we compute the twisted first homology group  $H_1(\mathcal{H}_g; H_1(\Sigma_g))$ , using the Wajnryb's presentation of the handlebody mapping class group  $\mathcal{H}_g$  to prove Theorem 1.1 in the case when  $g = 2, 3$ .

In Section 6, we prove Theorem 1.2 and also compute the twisted first homology groups of  $\mathcal{H}_g^*$  with coefficients in  $\text{Ker}(H_1(\Sigma_g) \rightarrow H_1(H_g))$  and  $H_1(H_g)$ .

## 2. ON THE SECOND HOMOLOGY OF THE HANDLEBODY MAPPING CLASS GROUPS FIXING A POINT OR A 2-DISK POINTWISE

In this section, we introduce some corollaries of Theorem 1.1 which give relationships between the second homology groups of  $\mathcal{H}_g$ ,  $\mathcal{H}_g^*$  and  $\mathcal{H}_{g,1}$ .

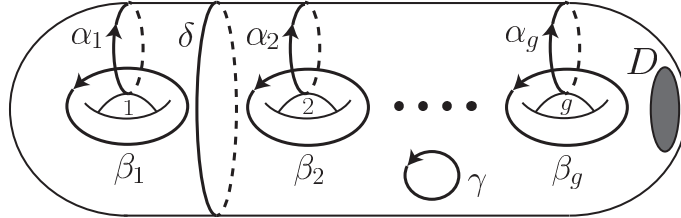


FIGURE 1. a 2-disk  $D$  and simple closed curves  $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g, \gamma$

Let  $U\Sigma_g$  denote the unit tangent bundle of  $\Sigma_g$ . Let  $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$  be oriented smooth simple closed curves as in figure 1, and denote their homology classes in  $H_1(\Sigma_g)$  by  $x_1 = [\alpha_1], x_2 = [\alpha_2], \dots, x_g = [\alpha_g], y_1 = [\beta_1], y_2 = [\beta_2], \dots, y_g = [\beta_g]$ . We also denote by  $\gamma$  a null-homotopic smooth simple closed curve in figure 1. There are natural liftings of  $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g, \gamma$  to  $U\Sigma_g$ , and let us denote their homology classes in  $H_1(U\Sigma_g)$  by  $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_g, \tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_g, z$ , respectively. For a group  $G$  and a  $G$ -module  $M$ , let us denote by  $M_G$  its coinvariant, that is, the quotient of  $M$  by the submodule spanned by the set  $\{gm - m \mid m \in M, g \in G\}$ .

**Lemma 2.1.** *For  $g \geq 2$ ,*

$$H_1(U\Sigma_g)_{\mathcal{H}_g} = 0.$$

*Proof.* For a simple closed curve  $c$  in  $\Sigma_g$ , we denote by  $t_c$  the Dehn twist along  $c$ . As in [6, Theorem 1B], we have  $t_{\alpha_i}(\tilde{y}_i) = \tilde{y}_i + \tilde{x}_i$  for  $i = 1, \dots, g$ . Note that our  $\tilde{c}$  is denoted by  $\tilde{c}$  in [6], and is different from  $\tilde{c}$ . Hence, we have  $\tilde{x}_1 = \dots = \tilde{x}_g = 0 \in H_1(U\Sigma_g)_{\mathcal{H}_g}$ . Let  $\delta'_i$  and  $\alpha'_i$  be simple closed curves as depicted in Figure 2 for  $1 \leq i \leq g-1$ . Let us denote  $h_i = t_{\delta'_i}^{-1} t_{\beta_i} t_{\alpha_{i+1}} \in \mathcal{M}_g$ . Since  $h_i(\alpha_l) = \alpha_l$  when  $l \neq i$ , and  $h_i(\alpha_i) = \alpha'_i$ , the mapping class  $h_i$  is actually an element of the handlebody mapping class group  $\mathcal{H}_g$ . We obtain

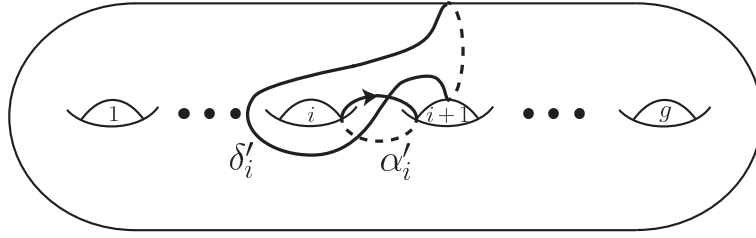


FIGURE 2. simple closed curves  $\delta'$  and  $\alpha'_i$

$$h_i(\tilde{x}_i) = \tilde{x}_i - \tilde{x}_{i+1} - z \text{ and } h_i(\tilde{y}_{i+1}) = \tilde{y}_i + \tilde{y}_{i+1} - z,$$

for  $i = 1, \dots, g-1$ . Thus we have  $z = \tilde{y}_1 = \dots = \tilde{y}_{g-1} = 0 \in H_1(U\Sigma_g)_{\mathcal{H}_g}$ . Since the rotation  $r \in \mathcal{H}_g$  of the surface  $\Sigma_g$  about a vertical line by 180 degrees maps  $\tilde{y}_g$  to  $-\tilde{y}_1$ , we also obtain  $\tilde{y}_g = 0$ .  $\square$

Using the mapping classes  $h_i$  and  $r$ , we can also show:

**Lemma 2.2.** *For  $g \geq 2$ ,*

$$\text{Ker}(H_1(\Sigma_g) \rightarrow H_1(H_g))_{\mathcal{H}_g} = 0.$$

**Proposition 2.3.** *When  $g \geq 4$ ,*

$$H_2(\mathcal{H}_g^*) \cong H_2(\mathcal{H}_g) \oplus \mathbb{Z}.$$

*Proof.* Let us denote the Lyndon-Hochschild-Serre spectral sequences of the forgetful exact sequences

$$1 \longrightarrow \pi_1 \Sigma_g \longrightarrow \mathcal{M}_g^* \longrightarrow \mathcal{M}_g \longrightarrow 1,$$

$$1 \longrightarrow \pi_1 \Sigma_g \longrightarrow \mathcal{H}_g^* \longrightarrow \mathcal{H}_g \longrightarrow 1$$

by  $\{E_{p,q}^r\}$  and  $\{\bar{E}_{p,q}^r\}$ , respectively. By Lemma 2.1, we have  $H_1(\Sigma_g)_{\mathcal{M}_g} = H_1(\Sigma_g)_{\mathcal{H}_g} = 0$ . Thus the  $E^\infty$  terms of both spectral sequences are as follows.

$E_{0,2}^\infty$	$*$	$*$	$\bar{E}_{0,2}^\infty$	$*$	$*$
0	$E_{1,1}^\infty$	$*$	0	$\bar{E}_{1,1}^\infty$	$*$
$\mathbb{Z}$	$H_1(\mathcal{M}_g)$	$H_2(\mathcal{M}_g)$	$\mathbb{Z}$	$H_1(\mathcal{H}_g)$	$H_2(\mathcal{H}_g)$

Therefore, we have a morphism of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bar{E}_{0,2}^\infty & \longrightarrow & \text{Ker}(H_2(\mathcal{H}_g^*) \rightarrow H_2(\mathcal{H}_g)) & \longrightarrow & \bar{E}_{1,1}^\infty \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & E_{0,2}^\infty & \longrightarrow & \text{Ker}(H_2(\mathcal{M}_g^*) \rightarrow H_2(\mathcal{M}_g)) & \longrightarrow & E_{1,1}^\infty \longrightarrow 0 \end{array}$$

induced by the inclusion  $\mathcal{H}_g^* \rightarrow \mathcal{M}_g^*$ . As explained in [9, Propositions 1.4 and 1.5],  $\text{Ker}(H_2(\mathcal{M}_g^*) \rightarrow H_2(\mathcal{M}_g)) \cong \mathbb{Z}$  and  $E_{0,2}^\infty = E_{0,2}^2 \cong \mathbb{Z}$  when  $g \geq 4$ . It is also true when  $g = 3$  as in [16, Corollary 4.9] (see also [14]). Moreover, there exists a surjective homomorphism  $S_1 : H_2(\mathcal{M}_g^*) \rightarrow \mathbb{Z}$  defined in [3, Section 0] which maps the fundamental class  $[\Sigma_g] \in H_2(\Sigma_g) = E_{0,2}^\infty$  to  $(2g-2)$ -times a generator and whose restriction to  $\text{Ker}(H_2(\mathcal{M}_g^*) \rightarrow H_2(\mathcal{M}_g))$  is surjective. These facts show that  $E_{1,1}^\infty$  is a cyclic group of order  $2g-2$ . Since Morita [12] showed  $E_{1,1}^2 = H_1(\mathcal{M}_g; H_1(\Sigma_g)) \cong \mathbb{Z}/(2g-2)\mathbb{Z}$  when  $g \geq 2$ , we obtain  $E_{1,1}^2 = E_{1,1}^\infty$ .

When  $g \geq 4$ , this fact and the isomorphism  $H_1(\mathcal{H}_g; H_1(\Sigma_g)) \cong H_1(\mathcal{M}_g; H_1(\Sigma_g))$  show that in the commutative diagram

$$\begin{array}{ccc} \bar{E}_{1,1}^2 & \longrightarrow & \bar{E}_{1,1}^\infty \\ \downarrow & & \downarrow \\ E_{1,1}^2 & \longrightarrow & E_{1,1}^\infty \end{array}$$

we have an isomorphism  $\bar{E}_{1,1}^\infty \cong E_{1,1}^\infty$ . As a conclusion, we obtain

$$\text{Ker}(H_2(\mathcal{H}_g^*) \rightarrow H_2(\mathcal{H}_g)) \cong \text{Ker}(H_2(\mathcal{M}_g^*) \rightarrow H_2(\mathcal{M}_g)) \cong \mathbb{Z}.$$

Consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & H_2(\mathcal{H}_g^*) & \longrightarrow & H_2(\mathcal{H}_g) \longrightarrow 0, \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & H_2(\mathcal{M}_g^*) & \longrightarrow & H_2(\mathcal{M}_g) \longrightarrow 0. \end{array}$$

Since the lower exact sequence splits, we obtain  $H_2(\mathcal{H}_g^*) \cong H_2(\mathcal{H}_g) \oplus \mathbb{Z}$ . □

When  $g \geq 2$ , Lemma 2.1 and the five term exact sequences induced by the exact sequences

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_1 \Sigma_g & \longrightarrow & \mathcal{H}_g^* & \longrightarrow & \mathcal{H}_g \longrightarrow 1, \\ 1 & \longrightarrow & \pi_1 U\Sigma_g & \longrightarrow & \mathcal{H}_{g,1} & \longrightarrow & \mathcal{H}_g \longrightarrow 1 \end{array}$$

imply:

**Lemma 2.4.** *When  $g \geq 2$ ,*

$$H_1(\mathcal{H}_{g,1}) \cong H_1(\mathcal{H}_g^*) \cong H_1(\mathcal{H}_g).$$

*Remark 2.5.* By the Wajnryb's presentation which we review in Section 4.1, we can compute the abelianization as follows:

$$H_1(\mathcal{H}_g) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } g = 1, \\ \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2 & \text{if } g = 2, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } g \geq 3. \end{cases}$$

We can also see that it is generated by  $s_1 = t_{\beta_1} t_{\alpha_1}^2 t_{\beta_1}$  when  $g \geq 3$ . Note that Wajnryb made a mistake in his calculation of the abelianization in [19, Theorem 20] when  $g = 2$ .

In the following, we choose a 2-disk  $D$  in the boundary  $\Sigma_g$  so that it is disjoint from the simple closed curves  $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$  as in Figure 1 and pick a point  $*$  in  $\text{Int } D$ .

**Lemma 2.6.** *When  $g \geq 3$ ,*

$$H_2(\mathcal{H}_g^*) \cong H_2(\mathcal{H}_{g,1}) \oplus \mathbb{Z}.$$

*Proof.* Let  $\pi : \mathcal{H}_{g,1} \rightarrow \mathcal{H}_g^*$  denote the forgetful map. The Gysin exact sequence of the central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{H}_{g,1} \xrightarrow{\pi} \mathcal{H}_g^* \longrightarrow 1.$$

is written as

$$H_1(\mathcal{H}_g^*) \xrightarrow{\pi^!} H_2(\mathcal{H}_{g,1}) \rightarrow H_2(\mathcal{H}_g^*) \rightarrow \mathbb{Z} \xrightarrow{\pi^!} H_1(\mathcal{H}_{g,1}).$$

Recall that the Gysin homomorphism  $\pi^! : H_1(\mathcal{H}_g^*) \rightarrow H_2(\mathcal{H}_{g,1})$  maps  $[h]$  to  $[\tilde{h}|t_{\partial D}] - [t_{\partial D}|\tilde{h}]$  in the bar resolution for  $g \in \mathcal{H}_g^*$ , where  $\tilde{h} \in \mathcal{H}_{g,1}$  is the inverse image of  $h$  under  $\pi$ . By Lemma 2.4 and [19, Theorem 20],  $H_1(\mathcal{H}_g^*)$  is the cyclic group of order 2 generated by  $s_1$  when  $g \geq 3$ . Note that we can choose a representing diffeomorphism of  $s_1$  whose support is in a genus 1 subsurface  $S$  of  $\Sigma_g - \text{Int } D$ . Moreover, using the lantern relation, we can obtain a 2-chain which bounds  $[t_{\partial D}] \in C_1(\mathcal{H}_{g,1})$  whose support is in  $(\Sigma_g - \text{Int } D) - S$ . Thus there exists a 3-chain which bounds  $[\tilde{s}_1|t_{\partial D}] - [t_{\partial D}|\tilde{s}_1] \in C_2(\mathcal{H}_{g,1})$ , where  $\tilde{s}_1 = t_{\beta_1} t_{\alpha_1}^2 t_{\beta_1} \in \mathcal{H}_{g,1}$ . Hence, the Gysin homomorphism  $\pi^! : H_1(\mathcal{H}_g^*) \rightarrow H_2(\mathcal{H}_{g,1})$  is the zero map. Since  $[t_{\partial D}] = 0 \in H_1(\mathcal{H}_{g,1})$ , the homomorphism  $\pi^! : \mathbb{Z} \rightarrow H_1(\mathcal{H}_{g,1})$  is also trivial. Thus we obtain the exact sequence

$$0 \rightarrow H_2(\mathcal{H}_{g,1}) \rightarrow H_2(\mathcal{H}_g^*) \rightarrow \mathbb{Z} \rightarrow 0.$$

□

Both of the direct sum decompositions of  $H_2(\mathcal{H}_g^*)$  in Proposition 2.3 and Lemma 2.6 are induced by the composition of the natural homomorphism  $H_2(\mathcal{H}_g^*) \rightarrow H_2(\mathcal{M}_g^*)$  and  $S_1 : H_2(\mathcal{M}_g^*) \rightarrow \mathbb{Z}$  defined in [3, Section 4] up to sign. Thus we obtain:

**Corollary 2.7.** *When  $g \geq 4$ ,*

$$H_2(\mathcal{H}_{g,1}) \cong H_2(\mathcal{H}_g).$$

### 3. PROOF OF THEOREM 1.1 FOR $g \geq 4$

In the rest of this paper, we write  $H$  for  $H_1(\Sigma_g)$  and denote by  $L$  the kernel of the homomorphism  $H_1(\Sigma_g) \rightarrow H_1(H_g)$  induced by the inclusion for simplicity. Note that  $H_1(H_g)$  is isomorphic to  $H/L$  as an  $\mathcal{H}_g$ -module. In this section, we prove Theorem 1.1 when  $g \geq 4$ . Luft's result on  $\text{Ker}(\mathcal{H}_g \rightarrow \text{Out } F_g)$  and Satoh's result on  $\text{Out } F_g$  make it much easier to determine the first homology  $H_1(\mathcal{H}_g; H)$  when  $g \geq 4$  than when  $g = 2, 3$ .

**Lemma 3.1.** *Let  $g \geq 2$ , and  $G$  a subgroup of the mapping class group  $\mathcal{M}_g$ . When the induced map  $H_1(U\Sigma_g)_G \rightarrow H_G$  by the natural projection is injective, there exists a surjective homomorphism*

$$H_1(G; H) \rightarrow \mathbb{Z}/(2g-2)\mathbb{Z}.$$

*Proof.* The exact sequence

$$0 \rightarrow \mathbb{Z}/(2g-2)\mathbb{Z} \rightarrow H_1(U\Sigma_g) \rightarrow H \rightarrow 0$$

induces the exact sequence

$$H_1(G; H) \rightarrow \mathbb{Z}/(2g-2)\mathbb{Z} \rightarrow H_1(U\Sigma_g)_G \rightarrow H_G.$$

Thus, we obtain the surjective homomorphism  $H_1(\mathcal{H}_g; H) \rightarrow \mathbb{Z}/(2g-2)\mathbb{Z}$ .  $\square$

*Remark 3.2.* The homomorphism  $H_1(G; H) \rightarrow \mathbb{Z}/(2g-2)\mathbb{Z}$  is written in [12, Section 6] explicitly. This coincide with the mod  $(2g-2)$ -reduction of the contraction of the twisted homomorphism called the first Johnson homomorphism. In particular, the homomorphism  $H_1(G; H) \rightarrow H_1(\mathcal{M}_g; H)$  induced by the inclusion is surjective. Note that the handlebody mapping class group  $\mathcal{H}_g$  satisfies the assumption of Lemma 3.1 because of Lemma 2.1.

By Lemma 2.1, we obtain a lower bound on the order of  $H_1(\mathcal{H}_g; H)$ . For a simple closed curve  $c$  in  $\Sigma_g$ , we denote by  $\mathcal{H}_g(c)$  the subgroup of  $\mathcal{H}_g$  which preserves the curve  $c$  setwise.

**Lemma 3.3.** *Let  $M$  be an  $\mathcal{H}_g$ -module on which  $\mathcal{L}_g$  acts trivially. Then, we have an exact sequence*

$$M_{\mathcal{H}_g(\alpha_1)} \longrightarrow H_1(\mathcal{H}_g; M) \longrightarrow H_1(\text{Out } F_g; M_{\mathcal{L}_g}) \longrightarrow 0.$$

*Proof.* The short exact sequence  $1 \rightarrow \mathcal{L}_g \rightarrow \mathcal{H}_g \rightarrow \text{Out } F_g \rightarrow 1$  induces an exact sequence

$$(3.1) \quad H_1(\mathcal{L}_g; M)_{\mathcal{H}_g} \rightarrow H_1(\mathcal{H}_g; M) \rightarrow H_1(\text{Out } F_g; M_{\mathcal{L}_g}) \rightarrow 0.$$

Luft [10, Corollary 2.4] proved that  $\mathcal{L}_g$  is normally generated by the Dehn twists along the curves  $\alpha_1$  and  $\delta$  in Figure 1. When  $g \geq 2$ , the lantern relation implies that the Dehn twist  $t_\delta$  can be written as a product of Dehn twists along boundary curves of meridian disks. Thus,  $\mathcal{L}_g$  is normally generated by the Dehn twist along  $\alpha_1$ . Since  $\mathcal{L}_g$  acts on  $M$  trivially, we have  $H_1(\mathcal{L}_g; M)_{\mathcal{H}_g} = (H_1(\mathcal{L}_g) \otimes M)_{\mathcal{H}_g}$ , and it is generated by  $\{t_{\alpha_1} \otimes m \mid m \in M\}$ . Since the surjective homomorphism  $M \rightarrow H_1(\mathcal{L}_g; M)_{\mathcal{H}_g}$  defined by  $m \mapsto t_{\alpha_1} \otimes m$  factors through  $M_{\mathcal{H}_g(\alpha_1)}$ , the exact sequence (3.1) and this homomorphism induce the desired exact sequence.  $\square$

**Lemma 3.4.** (1)  $H_1(\mathcal{L}_g; H/L)_{\mathcal{H}_g} \cong 0$  or  $\mathbb{Z}/2\mathbb{Z}$  when  $g \geq 2$ .

(2)  $H_1(\mathcal{L}_g; L)_{\mathcal{H}_g} = 0$  when  $g \geq 3$ , and  $H_1(\mathcal{L}_2; L)_{\mathcal{H}_2} \cong 0$  or  $\mathbb{Z}/2\mathbb{Z}$  when  $g = 2$ .

*Proof.* (1) Let us denote by  $\bar{y}_i$  the image of  $y_i$  under the natural homomorphism  $H \rightarrow H_1(H_g) \cong H/L$  induced by the inclusion. There exists a mapping class  $r_{1,j} \in \mathcal{H}_g$  for  $1 < j \leq g$  which preserves  $\alpha_1$  setwise and satisfies

$$r_{1,j}(x_l) = \begin{cases} -x_1 - x_2 - \cdots - x_j, & \text{if } l = j, \\ x_l, & \text{otherwise,} \end{cases} \quad r_{1,j}(y_l) = \begin{cases} y_l - y_j, & \text{if } 1 \leq l \leq j-1, \\ -y_j, & \text{if } l = j, \\ y_l, & \text{otherwise.} \end{cases}$$

See Lemma 4.3 for details. Then, we have  $r_{1,j}(t_{\alpha_1} \otimes \bar{y}_1) = t_{\alpha_1} \otimes \bar{y}_1 \in H_1(\mathcal{L}_g; H/L)_{\mathcal{H}_g}$ . Since  $r_{1,j}$  commutes with  $t_{\alpha_1}$ , and  $r_{1,j}(\bar{y}_1) = \bar{y}_1 - \bar{y}_j$ , we obtain  $t_{\alpha_1} \otimes [\bar{y}_j] = 0 \in H_1(\mathcal{L}_g; H/L)_{\mathcal{H}_g}$  for  $j = 2, \dots, g$ . Since the mapping class  $(t_{\beta_1} t_{\alpha_1})^3$  preserves each  $\alpha_i$  setwise for  $i = 1, 2, \dots, g$ , it is an element in  $\mathcal{H}_g$ . Since it satisfies  $(t_{\beta_1} t_{\alpha_1})^3(\bar{y}_1) = -\bar{y}_1$ , we have  $t_{\alpha_1} \otimes [2\bar{y}_1] = 0 \in H_1(\mathcal{L}_g; H/L)_{\mathcal{H}_g}$ . As a conclusion, we obtain  $H_1(\mathcal{L}_g; H/L)_{\mathcal{H}_g} = 0$  or  $\mathbb{Z}/2\mathbb{Z}$ .

(2) Since  $r_{1,j}$  commutes with  $t_{\alpha_1}$  for  $j = 2, 3, \dots, g$ , we obtain  $t_{\alpha_1} \otimes [x_1 + x_2 + \cdots + 2x_j] = 0 \in H_1(\mathcal{L}_g; L)_{\mathcal{H}_g}$ . For  $j = 1, 2, \dots, g$ , the mapping class  $s_j = t_{\beta_j} t_{\alpha_j}^2 t_{\beta_j} \in \mathcal{H}_g$  also preserves  $\alpha_1$  setwise, and satisfies

$$s_j(x_j) = -x_j.$$

Thus, we also have  $t_{\alpha_1} \otimes [2x_j] = 0 \in H_1(\mathcal{L}_g; L)_{\mathcal{H}_g}$ . Consequently, we obtain  $t_{\alpha_1} \otimes [x_1] = t_{\alpha_1} \otimes [x_2] = \cdots = t_{\alpha_1} \otimes [x_{g-1}] = t_{\alpha_1} \otimes [2x_g] = 0$ , and it implies  $H_1(\mathcal{L}_g; L)_{\mathcal{H}_g} \cong 0$  or  $\mathbb{Z}/2\mathbb{Z}$ .

Now suppose  $g \geq 3$ . Then, there exists a mapping class  $t_{g-1}$  (see Lemma 4.3) which preserves  $\alpha_1$  setwise and satisfies  $t_{g-1}(x_g) = x_{g-1}$ . Thus, we also obtain  $t_{\alpha_1} \otimes [x_g - x_{g-1}] = 0$  when  $g \geq 3$ , and it implies  $H_1(\mathcal{L}_g; L)_{\mathcal{H}_g} = 0$ .  $\square$

Applying Lemma 3.3 to the cases  $M = H/L$  and  $L$ , Lemma 3.4 implies:

**Lemma 3.5.** *When  $g \geq 3$ , the exact sequence  $1 \rightarrow \mathcal{L}_g \rightarrow \mathcal{H}_g \rightarrow \text{Out } F_g \rightarrow 1$  induces an isomorphism*

$$H_1(\mathcal{H}_g; L) \cong H_1(\text{Out } F_g; H^1(F_g)).$$

*When  $g \geq 2$ , it also induces an exact sequence*

$$\mathbb{Z}/2\mathbb{Z} \longrightarrow H_1(\mathcal{H}_g; H/L) \longrightarrow H_1(\text{Out } F_g; H_1(F_g)) \longrightarrow 0.$$

The twisted first homology groups of  $\text{Out } F_n$  with coefficients in  $H_1(F_n)$  and  $H^1(F_n)$  were computed by Satoh [17, Theorem 1 (2)] as follows.

**Theorem 3.6** (Satoh [17, Theorem 1 (2)]).

$$H_1(\text{Out } F_n; H^1(F_n)) \cong \begin{cases} \mathbb{Z}/(n-1)\mathbb{Z} & \text{when } n \geq 4, \\ (\mathbb{Z}/2\mathbb{Z})^2 & \text{when } n = 3, \\ \mathbb{Z}/2\mathbb{Z} & \text{when } n = 2, \end{cases}$$

$$H_1(\text{Out } F_n; H_1(F_n)) \cong \begin{cases} 0 & \text{when } n \geq 4, \\ \mathbb{Z}/2\mathbb{Z} & \text{when } n = 2, 3. \end{cases}$$



By Lemma 3.5 and Theorem 3.6, we obtain:

**Lemma 3.7.**

$$H_1(\mathcal{H}_g; L) \cong \begin{cases} \mathbb{Z}/(g-1)\mathbb{Z} & \text{if } g \geq 4, \\ (\mathbb{Z}/2\mathbb{Z})^2 & \text{if } g = 3, \end{cases} \quad H_1(\mathcal{H}_g; H/L) \cong \mathbb{Z}/2\mathbb{Z} \text{ if } g \geq 4.$$

*Remark 3.8.* Theorem 3.6 and Lemma 3.7 show  $\text{Ker}(H_1(\mathcal{H}_g; H/L) \rightarrow H_1(\text{Out } F_g; H_1(F_g))) \cong \mathbb{Z}/2\mathbb{Z}$  when  $g \geq 4$ . Thus Lemma 3.4 (1) implies  $H_1(\mathcal{L}_g; H/L)_{\mathcal{H}_g} \cong \mathbb{Z}/2\mathbb{Z}$  when  $g \geq 4$ .

*Remark 3.9.* By Lemma 3.5 and Theorem 3.6, we see that the order of  $H_1(\mathcal{H}_g; H/L)$  for  $g = 2, 3$  is at most 4. In Propositions 5.9 and 5.18, we will show  $H_1(\mathcal{H}_2; L) \cong \mathbb{Z}/2\mathbb{Z}$  and  $H_1(\mathcal{H}_g; H/L) \cong (\mathbb{Z}/2\mathbb{Z})^2$  for  $g = 2, 3$ . By Lemma 3.4 (1) and Theorem 3.6, it also follows that  $H_1(\mathcal{L}_g; H/L)_{\mathcal{H}_g} \cong \mathbb{Z}/2\mathbb{Z}$  for  $g = 2, 3$ .

By Lemma 2.2,  $H_0(\mathcal{H}_g; L) = L_{\mathcal{H}_g} = 0$  for  $g \geq 2$ . Thus, the short exact sequence of  $\mathcal{H}_g$ -modules  $0 \rightarrow L \rightarrow H \rightarrow H/L \rightarrow 0$  induces an exact sequence

$$(3.2) \quad H_1(\mathcal{H}_g; L) \longrightarrow H_1(\mathcal{H}_g; H) \longrightarrow H_1(\mathcal{H}_g; H/L) \longrightarrow 0.$$

Lemmas 3.1 and 3.7 and the exact sequence (3.2) give an upper bound on the order of  $H_1(\mathcal{H}_g; H)$ . Comparing this with the lower bound obtained in Lemma 3.1, we complete the proof of Theorem 1.1 for  $g \geq 4$ .

*Remark 3.10.* In the proof of Theorem 1.1 above, we also see the sequence

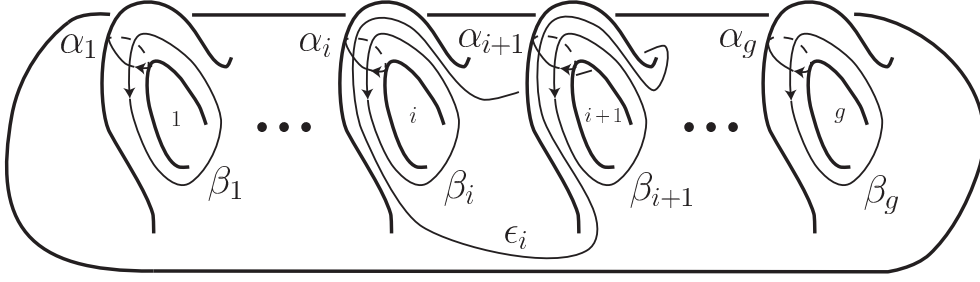
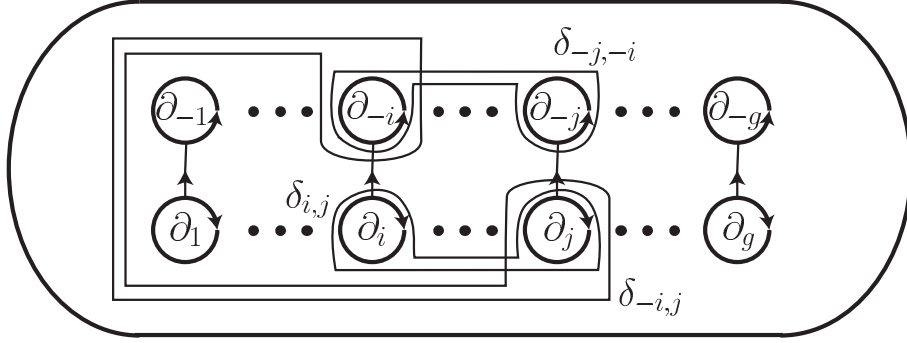
$$0 \longrightarrow H_1(\mathcal{H}_g; L) \longrightarrow H_1(\mathcal{H}_g; H) \longrightarrow H_1(\mathcal{H}_g; H/L) \longrightarrow 0$$

is exact when  $g \geq 4$ .

#### 4. THE WAJNRYB'S PRESENTATION OF THE HANDLEBODY MAPPING CLASS GROUP

In this section, we review the Wajnryb's presentation of the handlebody mapping class group  $\mathcal{H}_g$  and compute the action of the handlebody mapping class group  $\mathcal{H}_g$  to the first homology  $H_1(\Sigma_g)$ . This is for preparing to calculate the twisted first homology  $H_1(\mathcal{H}_g; H)$  when  $g = 2, 3$  in Section 5.

**4.1. A presentation of the handlebody mapping class group.** Let  $g \geq 2$ . We identify the surface in Figure 1 with that in Figure 3. Let  $\epsilon_i$  be a simple closed curve in Figure 3 for  $i = 1, \dots, g-1$ . By cutting the surface  $\Sigma_g$  along the simple closed curves  $\alpha_1, \dots, \alpha_g$ , we obtain a  $(2g)$ -holed sphere with boundary components  $\{\partial_{-i}, \partial_i\}_{i=1}^g$  as in Figure 4, where  $\alpha_i$  and  $\beta_i$  correspond to the boundary components  $\partial_{-i} \amalg \partial_i$  and the path from  $\partial_{-i}$  to  $\partial_i$ , respectively. For integers  $i, j$  satisfying  $1 \leq i < j \leq g$ , we denote by  $\delta_{-j, -i}$  and  $\delta_{i, j}$  the simple closed curves in Figure 4. For integers  $i, j$  satisfying  $1 \leq i \leq g$  and  $1 \leq j \leq g$ , we also denote

FIGURE 3. the surface  $\Sigma_g$ FIGURE 4. the  $(2g)$ -holed sphere

by  $\delta_{-i,j}$  the simple closed curve in Figure 4. For simplicity, we denote by  $a_i, b_i, e_i, d_{1,2}$  the Dehn twists along the curves  $\alpha_i, \beta_i, \epsilon_i, \delta_{1,2}$ , respectively. Let us denote

$$\begin{aligned} I_0 &= \{-g, -(g-1), \dots, -2, -1, 1, 2, \dots, g-1, g\}, \\ s_1 &= b_1 a_1^2 b_1, \\ t_i &= e_i a_i a_{i+1} e_i, \text{ for } i = 1, \dots, g-1. \end{aligned}$$

Since  $t_i$  permutes the simple closed curves  $\alpha_i$  and  $\alpha_{i+1}$ , and fixes other  $\alpha_j$ , we also have  $t_i \in \mathcal{H}_g$ . In the following, we denote  $\varphi * \psi = \varphi \psi \varphi^{-1}$  for  $\varphi, \psi \in \mathcal{H}_g$ . For  $i, j \in I_0$  satisfying  $i < j$ , we denote

$$\begin{aligned} d_{i,j} &= (t_{i-1} t_{i-2} \cdots t_1 t_{j-1} t_{j-2} \cdots t_2) * d_{1,2} \text{ if } i > 0, \\ d_{i,j} &= (t_{-i-1}^{-1} t_{-i-2}^{-1} \cdots t_1^{-1} s_1^{-1} t_{j-1} t_{j-2} \cdots t_2) * d_{1,2} \text{ if } i < 0 \text{ and } i+j > 0, \\ d_{i,j} &= (t_{-i-1}^{-1} t_{-i-2}^{-1} \cdots t_1^{-1} s_1^{-1} t_j t_{j-1} \cdots t_2) * d_{1,2} \text{ if } j > 0 \text{ and } i+j < 0, \\ d_{i,j} &= (t_{-j-1}^{-1} t_{-j-2}^{-1} \cdots t_1^{-1} t_{-i-1}^{-1} t_{-i-2}^{-1} \cdots t_2^{-1} s_1^{-1} t_1^{-1} s_1^{-1}) * d_{1,2} \text{ if } j < 0, \\ d_{i,j} &= (t_{j-1}^{-1} d_{j-1,j} t_{j-2}^{-1} d_{j-2,j-1} \cdots t_1^{-1} d_{1,2}) * (s_1^2 a_1^4), \text{ if } i+j = 0. \end{aligned}$$

Here,  $d_{i,j}$  is actually the Dehn twist along  $\delta_{i,j}$  in Figure 4 as explained in [19, p. 211]. However, to give a presentation of  $\mathcal{H}_g$  with a small generating set, we treat  $d_{i,j}$  as the products above.

We also denote

$$d_I = (d_{i_1, i_2} d_{i_1, i_3} \cdots d_{i_1, i_n} d_{i_2, i_3} \cdots d_{i_2, i_n} d_{i_3, i_4} \cdots d_{i_{n-1}, i_n}) (a_{i_1} \cdots a_{i_n})^{2-n},$$

$$\text{where } I = \{i_1, \dots, i_n\} \subset I_0 \text{ and } i_1 < \cdots < i_n,$$

$$c_{i,j} = d_I, \text{ where } I = \{k \in I_0 \mid i \leq k \leq j\} \text{ for } i \leq j.$$

Here,  $d_I$  and  $c_{i,j}$  are the Dehn twists along simple closed curves which enclose  $\{\partial_{i_1}, \dots, \partial_{i_n}\}$  and  $\{\partial_i, \dots, \partial_j\}$ , respectively. See [19, p. 211], for details. Let us denote

$$\tilde{I} = \{(i, j) \in I_0^2 \mid i = 1, 1 < j\} \cup \{(i, j) \in I_0^2 \mid i < 0, -i < j \leq g + i\},$$

and

$$r_{i,j} = b_j a_j c_{i,j} b_j, \text{ for } (i, j) \in \tilde{I},$$

$$k_j = a_j a_{j+1} t_j d_{j,j+1}^{-1} \text{ for } j = 1, \dots, g-1,$$

$$s_j = (k_{j-1} k_{j-2} \cdots k_1) * s_1 \text{ for } j = 2, \dots, g,$$

$$z = a_1 a_2 \cdots a_g s_1 t_1 t_2 \cdots t_{g-1} s_1 t_1 \cdots t_{g-2} s_1 \cdots s_1 t_1 s_1 d_I, \text{ where } I = \{1, \dots, g\},$$

$$z_j = k_{j-1} k_{j-2} \cdots k_{g+1-j} z \text{ for } j > \frac{g}{2}.$$

Here,  $r_{i,j}$  also lies in  $\mathcal{H}_g$  as is explained in [19, p. 211]. For  $\varphi, \psi \in \mathcal{H}_g$ , let us denote their commutator by  $[\varphi, \psi] = \varphi \psi \varphi^{-1} \psi^{-1}$ .

**Theorem 4.1.** [19, Theorem18] *The handlebody mapping class group of genus  $g$  admits the following presentation: The set of generators consists of  $a_1, \dots, a_g$ ,  $d_{1,2}$ ,  $s_1$ ,  $t_1, \dots, t_{g-1}$ , and  $r_{i,j}$  for  $(i, j) \in \tilde{I}$ . The set of defining relations is:*

(P1)  $[a_i, a_j] = 1$ ,  $[a_i, d_{j,k}] = 1$ , for all  $i, j, k \in I_0$ ,

(P2) Let  $i, j, r, s \in I_0$ .

(a)  $d_{r,s}^{-1} * d_{i,j} = d_{i,j}$  if  $r < s < i < j$  or  $i < r < s < j$ ,

(b)  $d_{r,i}^{-1} * d_{i,j} = d_{r,j} * d_{i,j}$  if  $r < i < j$ ,

(c)  $d_{i,s}^{-1} * d_{i,j} = (d_{i,j} d_{s,j}) * d_{i,j}$  if  $i < s < j$ ,

(d)  $d_{r,s}^{-1} * d_{i,j} = [d_{r,j}, d_{s,j}] * d_{i,j}$  if  $r < i < s < j$ ,

(P3)  $d_{I_0} = 1$ ,

(P4)  $d_{I_k} = a_{|k|}$  where  $I_k = I_0 - \{k\}$  for  $k \in I_0$ ,

(P5)  $t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$  for  $i = 1, \dots, g-2$ , and  $[t_i, t_j] = 1$  if  $1 \leq i < j-1 < g-1$ ,

(P6)  $t_i^2 = d_{i,i+1} d_{-i-1, -i} a_i^{-2} a_{i+1}^{-2}$  for  $i = 1, \dots, g-1$ ,

(P7)  $[s_1, a_i] = 1$  for  $i = 1, \dots, g$ ,  $t_i * a_i = a_{i+1}$  for  $i = 1, \dots, g-1$ ,  $[a_i, t_j] = 1$  for  $i, j \in I_0$  satisfying  $j \neq i, i-1$ , and  $[t_i, s_1] = 1$  for  $i = 2, \dots, g-1$ ,

(P8)  $[s_1, d_{2,3}] = 1$ ,  $[s_1, d_{-2,2}] = 1$ ,  $s_1 t_1 s_1 t_1 = t_1 s_1 t_1 s_1$ , and  $[t_i, d_{1,2}] = 1$  for  $i = 1, 3, \dots, g-1$ ,

(P9)  $r_{i,j}^2 = s_j c_{i,j} s_j c_{i,j}^{-1}$  for  $(i, j) \in \tilde{I}$ ,

(P10) Let  $(i, j) \in \tilde{I}$ .

(a)  $r_{i,j} * a_j = c_{i,j}$  and  $[r_{i,j}, a_k] = 1$  if  $k \neq j$ ,

(b)  $[r_{i,j}, t_k] = 1$  if  $k \neq |i|, j$  or  $k = i = 1 < j-1$ ,

- (c)  $[r_{i,j}, s_k] = 1$  if  $k < |i|$ ,  $j < k$  or  $k = -i$ ,
- (d)  $[r_{i,j}, d_{k,m}] = 1$  if  $k, m \in \{i, \dots, j-1\}$  or  $k, m \notin \{-j, i, i+1, \dots, j\}$ ,
- (e)  $[r_{i,j}, z_j] = 1$  if  $(i, j) = (1, g)$  or  $j = g+i$ ,
- (f)  $r_{i,j} * d_{i,j} = d_J$  where  $J = \{k \in I_0; i < k \leq j\}$ ,
- (g)  $r_{1,j} * d_{-j,1-j} = (t_{j-2}t_{j-3} \cdots t_1) * c_{-1,j}$ ,
- (h)  $r_{i,j} * d_{-j,1-j} = (t_{j-2}t_{j-3} \cdots t_{1-i}) * c_{i-1,j}$  if  $i < 0$  and  $j+i > 1$ ,
- (i)  $r_{i,j}^{-1} * d_{-j-1,-j} = s_{j+1}^{-1} * c_{i,j+1}$  if  $j < g$ ,
- (P11)  $r_{i,j} * t_{j-1} = t_{j-1}^{-1} * r_{i,j}$  if  $(i, j) \in \tilde{I}$  and  $-i+1 \neq j$ ,
- (P12) (a) Let  $h_2 = k_{j-1}^{-1}t_{j-2}^{-1}t_{j-3}^{-1} \cdots t_1^{-1}k_{j-1}k_{j-2} \cdots k_2$ .

$$r_{1,j} = s_j c_{1,j} s_j c_{1,j}^{-1} k_{j-1} a_j c_{1,j-2} t_{j-1} c_{1,j-1}^{-1} t_{j-1}^{-1} r_{1,j-1}^{-1} s_{j-1} h_2 r_{1,2}^{-1} h_2^{-1} k_{j-1}^{-1}$$

for  $3 \leq j \leq g$ .

- (b) Let  $h_3 = s_1 k_{j-1} k_{j-2} \cdots k_2$ .

$$r_{-1,j} = h_3 r_{1,2}^{-1} h_3^{-1} s_j r_{1,j}^{-1} c_{-1,j-1}^{-1} c_{1,j-1} a_1 s_j c_{-1,j} s_j c_{-1,j}^{-1}$$

for  $2 \leq j \leq g-1$ .

- (c) Let  $h_3 = s_{-i} t_{-1-i}^{-1} t_{-2-i}^{-1} \cdots t_1^{-1} k_{j-1} k_{j-2} \cdots k_3 k_2$ .

$$r_{i,j} = h_3 r_{1,2}^{-1} h_3^{-1} s_j r_{i+1,j}^{-1} c_{i,j-1}^{-1} c_{i+1,j-1} a_{-i} s_j c_{i,j} s_j c_{i,j}^{-1}$$

for  $i < -1$  and  $(i, j) \in \tilde{I}$ .

*Remark 4.2.* Note that there are some mistakes in the Wajnryb's presentation in [19]. The mapping class  $z_j$  is defined as the conjugation of  $z$  by  $k_{j-1}k_{j-2} \cdots k_{g+1-j}$  in [19]. However, as mentioned in [15], it should be defined as the product  $k_{j-1}k_{j-2} \cdots k_{g+1-j}z$ . In (P11), the condition  $-i+1 \neq j$  is needed. The relations of type (P11) are obtained in the situation when the pair of simple closed curves  $\partial_k$  and  $\partial_{-k}$  are separated by  $\gamma_{i,j}$  for  $k = j, j-1$  (see CASE 1 in [19, p.223]), and the equation  $r_{-(j-1),j} * t_{j-1} = t_{j-1}^{-1} * r_{-(j-1),j}$  in fact does not hold for any  $2 \leq j \leq g$ . We also erase the relation  $s_1^2 = d_{-1,1} a_1^{-4}$  in (P6) written in [19]. This is because we already defined  $d_{-1,1}$  as  $s_1^2 a_1^4$ .

**4.2. Action on the first homology  $H_1(\Sigma_g)$ .** Here, we compute the action of the handlebody mapping class group  $\mathcal{H}_g$  on the first homology  $H_1(\Sigma_g)$  of the boundary surface. Recall that  $x_1, \dots, x_g, y_1, \dots, y_g$  are the homology classes represented by the simple closed curves  $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$  in Figures 1 and 3.

**Lemma 4.3.** For  $1 \leq i \leq g$ ,

$$a_i(x_l) = x_l, \quad a_i(y_l) = \begin{cases} x_i + y_i & \text{if } l = i, \\ y_l & \text{otherwise,} \end{cases}$$

and

$$s_i(x_l) = \begin{cases} -x_i & \text{if } l = i, \\ x_l & \text{otherwise,} \end{cases} \quad s_i(y_l) = \begin{cases} 2x_i - y_i & \text{if } l = i, \\ y_l & \text{otherwise.} \end{cases}$$

For each  $i, j \in I_0$  such that  $i < j$ ,

$$d_{i,j}(x_l) = x_l, \quad d_{i,j}(y_l) = \begin{cases} \varepsilon(i)x_{|i|} + \varepsilon(j)x_{|j|} + y_l & \text{if } l = |i|, |j|, \\ y_l & \text{otherwise,} \end{cases}$$

where  $\varepsilon(i) = 1$  if  $i > 0$ , and  $\varepsilon(i) = -1$  if  $i < 0$ .

For  $1 \leq i \leq g-1$ ,

$$t_i(x_l) = \begin{cases} x_{i+1} & \text{if } l = i, \\ x_i & \text{if } l = i+1, \\ x_l & \text{otherwise,} \end{cases} \quad t_i(y_l) = \begin{cases} x_i + y_{i+1} & \text{if } l = i, \\ x_{i+1} + y_i & \text{if } l = i+1, \\ y_l & \text{otherwise,} \end{cases}$$

and

$$k_i(x_l) = \begin{cases} x_{i+1} & \text{if } l = i, \\ x_i & \text{if } l = i+1, \\ x_l & \text{otherwise,} \end{cases} \quad k_i(y_l) = \begin{cases} y_{i+1} & \text{if } l = i, \\ y_i & \text{if } l = i+1, \\ y_l & \text{otherwise.} \end{cases}$$

For  $1 < j \leq g$ ,

$$r_{1,j}(x_l) = \begin{cases} -x_1 - \cdots - x_j & \text{if } l = j, \\ x_l & \text{otherwise,} \end{cases}$$

$$r_{1,j}(y_l) = \begin{cases} x_1 + \cdots + x_j + y_l - y_j & \text{if } 1 \leq l \leq j-1, \\ x_1 + \cdots + x_{j-1} + 2x_j - y_j & \text{if } l = j, \\ y_l & \text{otherwise,} \end{cases}$$

and for  $(i, j) \in \tilde{I}$  such that  $i < 0$ ,

$$r_{i,j}(x_l) = \begin{cases} -x_{-i+1} - \cdots - x_j & \text{if } l = j, \\ x_l & \text{otherwise,} \end{cases}$$

$$r_{i,j}(y_l) = \begin{cases} x_{-i+1} + \cdots + x_j + y_l - y_j & \text{if } -i+1 \leq l \leq j-1, \\ x_{-i+1} + \cdots + x_{j-1} + 2x_j - y_j & \text{if } l = j, \\ y_l & \text{otherwise.} \end{cases}$$

*Proof.* The equations for the mapping classes  $a_i$  and  $d_{i,j}$  are obvious because  $a_i$  and  $d_{i,j}$  are Dehn twists along  $\alpha_i$  and  $\delta_{i,j}$  respectively. Similarly we have

$$b_i(x_l) = \begin{cases} x_i - y_i & \text{if } l = i, \\ y_l & \text{otherwise,} \end{cases} \quad b_i(y_l) = y_l,$$

for  $1 \leq i \leq g$  and

$$e_i(x_l) = \begin{cases} x_i - y_i + y_{i+1} & \text{if } l = i, \\ x_{i+1} + y_i - y_{i+1} & \text{if } l = i+1, \\ x_l & \text{otherwise,} \end{cases} \quad e_i(y_l) = y_l,$$

for  $1 \leq i \leq g-1$ .

Since  $t_i = e_i a_i a_{i+1} e_i$ ,

$$\begin{aligned} t_i(x_i) &= (e_i a_i a_{i+1})(x_i - y_i + y_{i+1}) = e_i(x_{i+1} - y_i + y_{i+1}) = x_{i+1}, \\ t_i(x_{i+1}) &= (e_i a_i a_{i+1})(x_{i+1} + y_i - y_{i+1}) = e_i(x_i + y_i - y_{i+1}) = x_i, \\ t_i(y_i) &= (e_i a_i a_{i+1})(y_i) = e_i(x_i + y_i) = x_i + y_{i+1}, \\ t_i(y_{i+1}) &= (e_i a_i a_{i+1})(y_{i+1}) = e_i(x_{i+1} + y_{i+1}) = x_{i+1} + y_i \end{aligned}$$

and  $t_i$  acts trivially on other  $x_l$ 's and  $y_l$ 's.

Since  $k_i = a_i a_{i+1} t_i d_{i,i+1}^{-1}$ ,

$$\begin{aligned} k_i(x_i) &= (a_i a_{i+1} t_i)(x_i) = (a_i a_{i+1})(x_{i+1}) = x_{i+1}, \\ k_i(x_{i+1}) &= (a_i a_{i+1} t_i)(x_{i+1}) = (a_i a_{i+1})(x_i) = x_i, \\ k_i(y_i) &= (a_i a_{i+1} t_i)(-x_i - x_{i+1} + y_i) = (a_i a_{i+1})(-x_{i+1} + y_{i+1}) = y_{i+1}, \\ k_i(y_{i+1}) &= (a_i a_{i+1} t_i)(-x_i - x_{i+1} + y_{i+1}) = (a_i a_{i+1})(-x_i + y_i) = y_i, \end{aligned}$$

and  $k_i$  acts trivially on other  $x_l$ 's and  $y_l$ 's.

Since  $s_1 = b_1 a_1^2 b_1$ ,

$$\begin{aligned} s_1(x_1) &= (b_1 a_1^2)(x_1 - y_1) = b_1(-x_1 - y_1) = -x_1, \\ s_1(y_1) &= (b_1 a_1^2)(y_1) = b_1(2x_1 + y_1) = 2x_1 - y_1, \end{aligned}$$

and  $s_1$  acts trivially on other  $x_l$ 's and  $y_l$ 's. The elements  $s_i$ 's are inductively defined by the recurrence relation  $s_{i+1} = k_i s_i k_i^{-1}$ . The element  $k_i$  replaces  $x_i$  and  $x_{i+1}$  with each other and  $y_i$  and  $y_{i+1}$  also. Hence the equation for  $s_i$  follows by induction.

Lastly, we verify the equations for  $r_{i,j}$ . In the case  $0 < i < j$ ,

$$c_{i,j}(x_l) = x_l, \quad c_{i,j}(y_l) = \begin{cases} x_i + \cdots + x_j + y_l & \text{if } i \leq l \leq j, \\ y_l & \text{otherwise.} \end{cases}$$

Since  $r_{i,j} = b_j a_j c_{i,j} b_j$ ,

$$\begin{aligned} r_{1,j}(x_j) &= (b_j a_j c_{1,j})(x_j - y_j) \\ &= (b_j a_j)(-x_1 - \cdots - x_{j-1} - y_j) \\ &= -x_1 - \cdots - x_j, \end{aligned}$$

and for  $1 \leq l \leq j$

$$\begin{aligned} r_{1,j}(y_l) &= (b_j a_j c_{1,j})(y_l) \\ &= (b_j a_j)(x_1 + \cdots + x_j + y_l) \\ &= \begin{cases} x_1 + \cdots + x_j + y_l - y_j & \text{if } 1 \leq l \leq j-1, \\ x_1 + \cdots + x_{j-1} + 2x_j - y_j & \text{if } l = j. \end{cases} \end{aligned}$$

The element  $r_{1,j}$  acts trivially on other  $x_l$ 's and  $y_l$ 's.

In the case  $(i, j) \in \tilde{I}$  and  $i < 0$ ,

$$c_{i,j}(x_l) = x_l, \quad c_{i,j}(y_l) = \begin{cases} x_{-i+1} + \cdots + x_j + y_l & \text{if } -i+1 \leq l \leq j, \\ y_l & \text{otherwise.} \end{cases}$$

Hence

$$\begin{aligned} r_{i,j}(x_j) &= (b_j a_j c_{i,j})(x_j - y_j) \\ &= (b_j a_j)(-x_{-i+1} - \cdots - x_{j-1} - y_j) \\ &= -x_{-i+1} - \cdots - x_j, \end{aligned}$$

and for  $-i+1 \leq l \leq j$

$$\begin{aligned} r_{i,j}(y_l) &= (b_j a_j c_{i,j})(y_l) \\ &= (b_j a_j)(x_{-i+1} + \cdots + x_j + y_l) \\ &= \begin{cases} x_{-i+1} + \cdots + x_j + y_l - y_j & \text{if } -i+1 \leq l \leq j-1, \\ x_{-i+1} + \cdots + x_{j-1} + 2x_j - y_j & \text{if } l = j. \end{cases} \end{aligned}$$

The element  $r_{i,j}$  acts trivially on other  $x_l$ 's and  $y_l$ 's. □

## 5. PROOF OF THEOREM 1.1 FOR $g = 2, 3$

In this section, we prove Theorem 1.1 for  $g = 2, 3$ . We denote by  $A$  the ring  $\mathbb{Z}$  or  $\mathbb{Z}/n\mathbb{Z}$  for an integer  $n \geq 2$ , and  $H_A = H_1(\Sigma_g; A)$ . Recall that, for a group  $G$  and a left  $G$ -module  $M$ , a map  $d : G \rightarrow M$  is called a crossed homomorphism if it satisfies  $d(hh') = d(h) + hd(h')$  for  $h, h' \in G$ . For a group  $G$  and a left  $G$ -module  $M$ , we consider group cohomology  $H^*(G; M)$  as that of the standard chain complex induced by the bar resolution. Then, the space of 1-cocycles  $Z^1(G; M)$  is identified with the space of crossed homomorphisms from  $G$  to  $M$ , and the space of 1-coboundaries  $B^1(G; M)$  is identified with the image of the coboundary map  $\delta : M \rightarrow Z^1(G; M)$  defined by  $\delta(m)(h) = hm - m$  for  $m \in M$ . See [2, Section 2.3] for details.

As written in [19, Theorem 19], the handlebody mapping class group  $\mathcal{H}_g$  is generated by  $a_1, s_1, r_{1,2}, t_1$ , and  $u = t_1 t_2 \cdots t_{g-1}$ . Therefore, crossed homomorphisms  $d : \mathcal{H}_g \rightarrow H_A$  are uniquely determined by the values  $d(a_1), d(s_1), d(r_{1,2}), d(t_1)$ , and  $d(u)$ . Moreover, a 5-tuple of elements in  $H_A$  becomes values of  $a_1, s_1, r_{1,2}, t_1$ , and  $u$  under some crossed homomorphism  $d$  on  $\mathcal{H}_g$  if and only if they are compatible with the relations (P1)-(P12) in Theorem 4.1. The basis  $\{x_1, \dots, x_g, y_1, \dots, y_g\}$  of  $H_A$  induces an isomorphism  $H_A \cong A^{2g}$ . For  $v \in H_A$ , we denote its projection to the  $i$ -th coordinate of  $A^{2g}$  by  $v_i \in A$  for  $i = 1, 2, \dots, 2g$ .

**Lemma 5.1.**

$$H^1(\mathcal{H}_2; H_A) \cong \{d \in Z^1(\mathcal{H}_2; H_A); d(r_{1,2})_1 = d(s_1)_3 - d(r_{1,2})_4 = d(u)_2 = d(u)_4 = 0\},$$

$$H^1(\mathcal{H}_3; H_A) \cong \{d \in Z^1(\mathcal{H}_3; H_A);$$

$$d(r_{1,2})_1 = d(s_1)_4 - d(r_{1,2})_5 = d(u)_2 = d(u)_3 = d(u)_5 = d(u)_6 = 0\}.$$

*Proof.* Let  $f_2: Z^1(\mathcal{H}_2; H_A) \rightarrow A^4$  and  $f_3: Z^1(\mathcal{H}_3; H_A) \rightarrow A^6$  be homomorphisms defined by

$$\begin{aligned} f_2(d) &= (d(r_{1,2})_1, d(s_1)_3 - d(r_{1,2})_4, d(u)_2, d(u)_4), \text{ and} \\ f_3(d) &= (d(r_{1,2})_1, d(s_1)_4 - d(r_{1,2})_5, d(u)_2, d(u)_3, d(u)_5, d(u)_6), \end{aligned}$$

respectively. Then, the composition maps  $f_g \circ \delta: H_A \rightarrow A^{2g}$  are written as

$$\begin{aligned} f_2 \circ \delta(v) &= (-v_2 + v_3 + v_4, -v_3 + 2v_4, v_1 - v_2 + v_4, v_3 - v_4), \\ f_3 \circ \delta(v) &= (-v_2 + v_4 + v_5, -v_4 + 2v_5, v_1 - v_2 + v_5 + v_6, v_2 - v_3 + v_6, v_4 - v_5, v_5 - v_6), \end{aligned}$$

for  $v \in H_A$ . Since these maps are isomorphisms, we have

$$H^1(\mathcal{H}_g; H_A) = Z^1(\mathcal{H}_g; H_A) / B^1(\mathcal{H}_g; H_A) \cong \text{Ker } f_g$$

for  $g = 2, 3$ . □

**Lemma 5.2.** *Suppose  $d \in Z^1(\mathcal{H}_g; H_A)$  satisfies  $d(u)_2 = \cdots = d(u)_g = d(u)_{g+2} = \cdots = d(u)_{2g} = 0$  as in Lemma 5.1. Then,*

- (1)  $d(a_i) = u^{i-1}d(a_1)$ .
- (2)  $d(t_i) = u^{i-1}d(t_1)$ .

*Proof.* Note that  $a_i = u^{i-1}a_1u^{-(i-1)}$ . It can be checked using the relation (P7). Hence we have

$$d(a_{i+1}) = d(u) + ud(a_i) - ua_iu^{-1}d(u) = d(u) + ud(a_i) - a_{i+1}d(u).$$

Since  $(a_{i+1}v)_1 = v_1$  and  $(a_{i+1}v)_{g+1} = v_{g+1}$  for any  $v \in H_A$ , we have  $a_{i+1}d(u) = d(u)$ , and thus  $d(a_{i+1}) = ud(a_i)$ . By induction on  $i$ , we have the equation (1). The equation (2) can be similarly verified. □

**Lemma 5.3.** *Suppose  $d \in Z^1(\mathcal{H}_g; H_A)$  satisfies  $d(u)_2 = \cdots = d(u)_g = d(u)_{g+2} = \cdots = d(u)_{2g} = 0$  as in Lemma 5.1. Then*

- (1)  $d(a_1)_{g+1} = \cdots = d(a_1)_{2g} = 0$ .
- (2)  $2d(a_1)_2 = \cdots = 2d(a_1)_g = 0$ .
- (3)  $d(s_1)_{g+2} = \cdots = d(s_1)_{2g} = 0$ .
- (4)  $d(s_1)_{g+1} = -2d(a_1)_1$ .
- (5)  $d(a_1)_2 + d(r_{1,2})_{g+1} = 0$ .

*Proof.* For any  $i$  and  $j$ ,

$$d(a_i a_j) = d(a_i) + a_i d(a_j) = d(a_i) + d(a_j) + d(a_j)_{g+i} x_i.$$

Since  $a_1$  and  $a_i$  commute for any  $1 \leq i \leq g$  by the relation (P1), it must be  $d(a_1 a_i) = d(a_i a_1)$ , and thus  $d(a_1)_{g+i} = 0$  for any  $2 \leq i \leq g$ .

Since  $a_1$  and  $r_{1,2}$  commute by the relation (P10)(a), it must be

$$(1 - a_1)d(r_{1,2}) = (1 - r_{1,2})d(a_1).$$



Since  $((1 - r_{1,2})v)_{g+2} = v_{g+1} + 2v_{g+2}$  for any  $v \in H_A$  while  $((1 - a_1)v)_{g+2} = 0$ , we have  $d(a_1)_{g+1} = 0$  and thus the equation (1). Since  $((1 - r_{1,2})v)_1 = v_2 - v_{g+1} - v_{g+2}$  for any  $v \in H_A$  while  $((1 - a_1)v)_1 = -v_{g+1}$ , we have the equation (5).

Note that  $a_i$  and  $s_j$  commute for any  $1 \leq i, j \leq g$ . It can be verified using the relations (P1) and (P7). Hence it must be

$$(1 - s_j)d(a_i) = (1 - a_i)d(s_j).$$

Suppose  $i = 1$  and  $2 \leq j \leq g$ . Then we have the equation (2) because  $((1 - s_j)v)_j = 2v_j - 2v_{g+j}$  and  $((1 - a_1)v)_j = 0$  for any  $v \in H_A$ . Suppose  $2 \leq i \leq g$  and  $j = 1$ . Then we have the equation (3) because  $((1 - s_1)v)_i = 0$  and  $((1 - a_i)v)_i = -v_{g+i}$  for any  $v \in H_A$ . Suppose  $i = j = 1$ . Then we have the equation (4) because  $((1 - s_1)v)_1 = 2v_1 - 2v_{g+1}$  and  $((1 - a_1)v)_1 = -v_{g+1}$  for any  $v \in H_A$ .  $\square$

5.1.  $H^1(\mathcal{H}_2; H_A)$ . Here, we assume  $g = 2$  and prove that  $H^1(\mathcal{H}_2; H_A) \cong \text{Hom}((\mathbb{Z}/2\mathbb{Z})^2, A)$ . Then, the universal coefficient theorem implies  $H_1(\mathcal{H}_2; H) \cong (\mathbb{Z}/2\mathbb{Z})^2$  and we complete the proof of Theorem 1.1 when  $g = 2$ .

Let  $d \in Z^1(\mathcal{H}_2; H_A)$  be a crossed homomorphism satisfying the condition  $d(r_{1,2})_1 = d(s_1)_3 - d(r_{1,2})_4 = d(u)_2 = d(u)_4 = 0$  as in Lemma 5.1. Note that in this case  $u = t_1$ . By Lemma 5.3, we can set

$$\begin{aligned} d(a_1) &= w_{1,1}x_1 + w_{1,2}x_2, \\ d(s_1) &= w_{2,1}x_1 + w_{2,2}x_2 + w_{2,3}y_1, \\ d(t_1) &= w_{3,1}x_1 + w_{3,3}y_1, \\ d(r_{1,2}) &= w_{4,2}x_2 + w_{4,3}y_1 + w_{4,4}y_2. \end{aligned}$$

By the condition on  $d$  and Lemma 5.3, we also have

$$(5.1) \quad 2w_{1,2} = 0, w_{2,3} = w_{4,4} = -2w_{1,1}, \text{ and } w_{1,2} + w_{4,3} = 0.$$

**Lemma 5.4.**

$$w_{1,2} = w_{4,3} = 0.$$

Moreover, we have

$$d(d_{1,2}) = w_{1,1}(x_1 + x_2).$$

*Proof.* Since  $w_{1,2} + w_{4,3} = 0$ , it suffices to prove that  $w_{4,3} = 0$ . Note that by Lemma 5.2, we have  $d(a_2) = t_1 d(a_1) = w_{1,2}x_1 + w_{1,1}x_2$ . Since  $d_{1,2} = r_{1,2}a_2r_{1,2}^{-1}$  by the relation (P10)(a),

$$\begin{aligned} d(d_{1,2}) &= d(r_{1,2}) + r_{1,2}d(a_2) - r_{1,2}a_2r_{1,2}^{-1}d(r_{1,2}) \\ &= (1 - d_{1,2})d(r_{1,2}) + r_{1,2}d(a_2) \\ &= (-w_{1,1} - w_{4,3} - w_{4,4})(x_1 + x_2) + w_{1,2}x_1. \end{aligned}$$

Since  $a_2 = r_{1,2}d_{1,2}r_{1,2}^{-1}$  by the relation (P10)(f),

$$d(a_2) = (1 - a_2)d(r_{1,2}) + r_{1,2}d(d_{1,2}) = w_{1,2}x_1 + (w_{1,1} + w_{4,3})x_2.$$

Thus we obtain  $w_{4,3} = 0$ . By the equation  $w_{4,4} = -2w_{1,1}$  in (5.1), we have  $d(d_{1,2}) = w_{1,1}(x_1 + x_2)$ .  $\square$

**Lemma 5.5.**

$$w_{2,3} = w_{3,1} = w_{3,3} = w_{4,4} = 0.$$

*In particular, we have  $d(t_1) = 0$  and  $2w_{1,1} = 0$ .*

*Proof.* Recall that  $d_{-2,-1} = (s_1 t_1 s_1)^{-1} d_{1,2} (s_1 t_1 s_1)$ . In the case  $g = 2$ , the Dehn twist  $d_{-2,-1}$  coincides with  $d_{1,2}$ . Hence the elements  $d_{1,2}$  and  $s_1 t_1 s_1$  commute and it must be  $(1 - d_{1,2})d(s_1 t_1 s_1) = (1 - s_1 t_1 s_1)d(d_{1,2})$ . Since

$$\begin{aligned} ((1 - d_{1,2})d(s_1 t_1 s_1))_1 &= ((1 - d_{1,2})d(s_1 t_1 s_1))_2 = -2w_{2,3} + w_{3,3}, \text{ while} \\ ((1 - s_1 t_1 s_1)d(d_{1,2}))_1 &= ((1 - s_1 t_1 s_1)d(d_{1,2}))_2 = 2w_{1,1}, \end{aligned}$$

we have  $2w_{1,1} = -2w_{2,3} + w_{3,3}$ . The equation  $w_{2,3} = -2w_{1,1}$  in (5.1) shows  $w_{2,3} = w_{3,3}$ .

Since  $w_{2,3} = w_{3,3} = w_{4,4}$ , it remains to prove that  $w_{3,1} = w_{3,3} = 0$ . Note that each of  $d(a_1)$ ,  $d(a_2)$ , and  $d(d_{1,2})$  are in  $L_A = \text{Ker}(H_1(\Sigma_2; A) \rightarrow H_1(H_2; A))$ . By the relation (P6),

$$t_1^2 = d_{1,2}^2 a_1^{-2} a_2^{-2}.$$

Since each of  $a_1$ ,  $a_2$  and  $d_{1,2}$  acts on  $L$  trivially, we have

$$d(d_{1,2}^2 a_1^{-2} a_2^{-2}) = 2(d(d_{1,2}) - d(a_1) - d(a_2)) = 0.$$

On the other hand, we have

$$d(t_1^2) = d(t_1) + t_1 d(t_1) = w_{3,1}(x_1 + x_2) + w_{3,3}(x_3 + x_4) + w_{3,3}x_1.$$

These equations show  $w_{3,1} = w_{3,3} = 0$ .  $\square$

**Lemma 5.6.**

$$w_{4,2} = 0.$$

*In particular, we have  $d(r_{1,2}) = 0$ .*

*Proof.* The relation  $t_1 r_{1,2} t_1 = r_{1,2} t_1 r_{1,2}$  in (P11) shows

$$(1 - t_1 + r_{1,2} t_1) d(r_{1,2}) = (1 - r_{1,2} + t_1 r_{1,2}) d(t_1).$$

By Lemma 5.5, the right hand side is equal to zero. Since

$$(1 - t_1 + r_{1,2} t_1) d(r_{1,2}) = w_{4,2} x_2,$$

we have  $w_{4,2} = 0$ .  $\square$

**Lemma 5.7.**

$$2w_{2,2} = 0.$$

*Proof.* The relation  $r_{1,2}^2 = s_2 d_{1,2} s_2 d_{1,2}^{-1}$  in (P9) and Lemma 5.6 show

$$d(s_2 d_{1,2} s_2 d_{1,2}^{-1}) = d(r_{1,2}^2) = 0.$$

Recall that  $k_1 = a_1 a_2 t_1 d_{1,2}^{-1}$  and  $s_2 = k_1 s_1 k_1^{-1}$ . Hence we have

$$\begin{aligned} d(k_1) &= d(a_1) + d(a_2) - t_1 d(d_{1,2}) = 0, \text{ and} \\ d(s_2) &= d(k_1) + k_1 d(s_1) - s_2 d(k_1) = k_1 d(s_1). \end{aligned}$$

Therefore, we have

$$d(s_2 d_{1,2} s_2 d_{1,2}^{-1}) = (1 + s_2 d_{1,2}) d(s_2) + (s_2 - 1) d(d_{1,2}) = 2w_{2,2} x_1$$

and thus  $2w_{2,2} = 0$ . □

**Lemma 5.8.**

$$w_{2,1} = w_{2,2}.$$

*Proof.* Recall that  $z = a_1 a_2 s_1 t_1 s_1 d_{1,2}$  and  $z_2 = k_1 z$ . Hence we have

$$\begin{aligned} d(z) &= d(a_1) + d(a_2) + (1 + s_1 t_1) d(s_1) + s_1 t_1 s_1 d(d_{1,2}) \\ &= (w_{2,1} + w_{2,2})(x_1 + x_2), \text{ and} \\ d(z_2) &= d(k_1) + k_1 d(z) = d(z). \end{aligned}$$

Hence

$$(1 - r_{1,2}) d(z_2) = (w_{2,1} + w_{2,2})(x_1 + 2x_2)$$

Since  $r_{1,2}$  and  $z_2$  commute by the relation (P10)(e), it must be  $(1 - r_{1,2}) d(z_2) = (1 - z_2) d(r_{1,2}) = 0$ . Thus we have  $w_{2,1} = w_{2,2}$ . □

Summarizing Lemmas 5.4, 5.5, 5.6, 5.7 and 5.8, we have

$$d(a_1) = w_{1,1} x_1, d(s_1) = w_{2,1}(x_1 + x_2) \text{ and } d(t_1) = d(r_{1,2}) = 0,$$

where  $2w_{1,1} = 2w_{2,1} = 0$ . It can be verified that such  $d$  is compatible with the relations (P1)–(P12). Now we have

$$H^1(\mathcal{H}_2; H_A) \cong \text{Ker } f_2 \cong \{(w_{1,1}, w_{2,1}) \in A^2; 2w_{1,1} = 2w_{2,1} = 0\}.$$

**Proposition 5.9.**

$$H_1(\mathcal{H}_2; L) \cong \mathbb{Z}/2\mathbb{Z}, H_1(\mathcal{H}_2; H/L) \cong H_1(\mathcal{H}_2; H),$$

and the homomorphism  $H_2(\mathcal{H}_2; H/L) \rightarrow H_1(\mathcal{H}_2; L)$  induced by the exact sequence  $0 \rightarrow L \rightarrow H \rightarrow H/L \rightarrow 0$  is surjective.

*Proof.* As well as Lemma 5.3, we can verify that

$$H^1(\mathcal{H}_2; H_A/L_A) \cong \{d' \in Z^1(\mathcal{H}_2; H_A/L_A); d'(s_1)_1 - d'(r_{1,2})_2 = d'(u)_2 = 0\}.$$

Similar calculations in Section 5.1 show that a crossed homomorphism  $d': \mathcal{H}_2 \rightarrow H_A/L_A$  such that  $d'(s_1)_1 - d'(r_{1,2})_2 = d'(u)_2 = 0$  is compatible with the relations (P1)–(P12) if and only if

$$d'(a_1) = d'(t_1) = 0, d'(s_1) = w'_{2,3}(y_1 + y_2) \text{ and } d'(r_{1,2}) = w'_{2,3}y_2$$

where  $w'_{2,3} \in A$  satisfies  $2w'_{2,3} = 0$ . Thus, we obtain  $H^1(\mathcal{H}_2; H_A/L_A) \cong \{w'_{2,3} \in A; 2w'_{2,3} = 0\}$ , and the universal coefficient theorem implies  $H_1(\mathcal{H}_2; L) \cong \mathbb{Z}/2\mathbb{Z}$ .

Next, we prove that  $H_1(\mathcal{H}_2; H/L) \cong H_1(\mathcal{H}_2; H)$  and the homomorphism  $H_2(\mathcal{H}_2; H/L) \rightarrow H_1(\mathcal{H}_2; L)$  is surjective. Since  $H_0(\mathcal{H}_2; L) = L_{\mathcal{H}_2} = 0$  as shown in Lemma 2.2, we have the exact sequence

$$H_2(\mathcal{H}_2; H/L) \longrightarrow H_1(\mathcal{H}_2; L) \longrightarrow H_1(\mathcal{H}_2; H) \longrightarrow H_1(\mathcal{H}_2; H/L) \longrightarrow 0.$$

Thus, it suffices to show that the homomorphism  $H_1(\mathcal{H}_2; L) \rightarrow H_1(\mathcal{H}_2; H)$  is the zero map.

As we saw in the proof of Theorem 1.1,  $\text{Ker}(f_2: Z^1(\mathcal{H}_2; H_A) \rightarrow A^4)$  is contained in the image of the homomorphism  $Z^1(\mathcal{H}_2; L_A) \rightarrow Z^1(\mathcal{H}_2; H_A)$ . Thus, the homomorphism  $H^1(\mathcal{H}_2; H_A) \rightarrow H^1(\mathcal{H}_2; H_A/L_A)$  induced by the projection  $H_A \rightarrow H_A/L_A$  is the zero map. The universal coefficient theorem implies  $H_1(\mathcal{H}_2; L) \rightarrow H_1(\mathcal{H}_2; H)$  is the zero map.  $\square$

5.2.  $H^1(\mathcal{H}_3; H_A)$ . Here, we assume  $g = 3$  and prove that  $H_1(\mathcal{H}_3; H_A) \cong \text{Hom}(\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, A)$ . Then, the universal coefficient theorem implies  $H_1(\mathcal{H}_3; H) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ , and we complete the proof of Theorem 1.1 when  $g = 3$ .

Let  $d \in Z^1(\mathcal{H}_3; H_A)$  be a crossed homomorphism satisfying the condition  $d(r_{1,2})_1 = d(s_1)_4 - d(r_{1,2})_5 = d(u)_2 = d(u)_3 = d(u)_5 = d(u)_6 = 0$  as in Lemma 5.1. By Lemma 5.3, we can set

$$\begin{aligned} d(a_1) &= w_{1,1}x_1 + w_{1,2}x_2 + w_{1,3}x_3, \\ d(s_1) &= w_{2,1}x_1 + w_{2,2}x_2 + w_{2,3}x_3 + w_{2,4}y_1, \\ d(t_1) &= w_{3,1}x_1 + w_{3,2}x_2 + w_{3,3}x_3 + w_{3,4}y_1 + w_{3,5}y_2 + w_{3,6}y_3, \\ d(r_{1,2}) &= w_{4,2}x_2 + w_{4,3}x_3 + w_{4,4}y_1 + w_{4,5}y_2 + w_{4,6}y_3. \end{aligned}$$

By the condition on  $d$  and Lemma 5.3, we also have

$$(5.2) \quad 2w_{1,2} = 2w_{1,3} = 0, w_{2,4} = w_{4,5} = -2w_{1,1}, \text{ and } w_{1,2} + w_{4,4} = 0.$$

**Lemma 5.10.** (1)  $w_{1,2} = w_{4,4} = 0$ .

$$(2) \quad w_{1,3} = 0.$$

$$(3) \quad 2w_{3,3} = 0.$$

$$(4) \quad w_{3,4} + w_{3,5} = 0.$$

$$(5) \quad w_{3,6} = 0.$$

$$(6) \quad d(d_{1,2}) = w_{1,2}(x_1 + x_2).$$

*Proof.* Since  $a_1$  and  $r_{1,3}$  commute by the relation (P10)(a), it must be  $(1 - a_1)d(r_{1,3}) = (1 - r_{1,3})d(a_1)$ . Since

$$((1 - r_{1,3})d(a_1))_2 = w_{1,3}, \text{ while } (1 - a_1)d(r_{1,3})_2 = 0,$$

we have the equation (2).

Since  $d(t_2) = ud(t_1) = t_1t_2d(t_1)$  by Lemma 5.2, we have

$$\begin{aligned} d(t_2) &= (w_{3,3} + w_{3,4} + w_{3,5})x_1 + (w_{3,1} + w_{3,6})x_2 + (w_{3,2} + w_{3,6})x_3 \\ &\quad + w_{3,6}y_1 + w_{3,4}y_2 + w_{3,5}y_3. \end{aligned}$$

Since  $a_1$  and  $t_2$  commute by the relation (P7), it must be  $(1 - a_1)d(t_2) = (1 - t_2)d(a_1)$ . Since

$$((1 - t_2)d(a_1))_2 = w_{1,2} \text{ while } (1 - a_1)d(t_2)_2 = 0,$$

we have the equation (1). Furthermore, since

$$(1 - a_1)d(t_2)_1 = -w_{3,6} \text{ while } ((1 - t_2)d(a_1))_1 = 0,$$

we have the equation (5).

Now we have  $d(a_1) = w_{1,1}x_1$ . Note that by Lemma 5.2,  $d(a_i) = u^{i-1}d(a_1) = w_{1,1}x_i$  for any  $1 \leq i \leq 3$ . Since  $d_{1,2} = r_{1,2}a_2r_{1,2}^{-1}$  by the relation (P10)(a), we have

$$d(d_{1,2}) = (1 - d_{1,2})d(r_{1,2}) + r_{1,2}d(a_2) = (-w_{1,1} - w_{4,5})(x_1 + x_2).$$

Since  $w_{4,5} = -2w_{1,1}$ , we have the equation (6).

Since  $d_{1,2}$  and  $t_1$  commute by the relation (P8), it must be  $(1 - d_{1,2})d(t_1) = (1 - t_1)d(d_{1,2}) = 0$ . Since  $(1 - d_{1,2})d(t_1) = -(w_{3,4} + w_{3,5})(x_1 + x_2)$ , we have the equation (4).

Since  $s_1$  and  $t_2$  commute by the relation (P7), it must be  $(1 - s_1)d(t_2) = (1 - t_2)d(s_1)$ . Since  $((1 - s_1)d(t_2))_1 = 2w_{3,3}$  while  $((1 - t_2)d(t_1))_1 = 0$ , we have the equation (3).  $\square$

Since  $((1 - t_2)d(s_1))_2 = w_{2,2} - w_{2,3} - w_{3,4}$  while  $(1 - s_1)d(t_2)_2 = 0$ , we have

$$(5.3) \quad w_{2,2} - w_{2,3} - w_{3,4} = 0.$$

**Lemma 5.11.**

$$w_{3,2} = w_{3,3} \text{ and } w_{3,4} = w_{3,5} = 0.$$

*Proof.* Since  $d(u) = d(t_1t_2) = d(t_1) + t_1d(t_2)$ , a straightforward computation shows

$$d(u)_2 = w_{3,2} + w_{3,3} + w_{3,4}, d(u)_3 = w_{3,2} + w_{3,3}, \text{ and } d(u)_5 = d(u)_6 = w_{3,5}.$$

Since  $d(u)_2, d(u)_3, d(u)_5, d(u)_6 = 0$ , we have

$$w_{3,2} + w_{3,3} = w_{3,4} = w_{3,5} = 0.$$

Lemma 5.10 (3) implies  $w_{3,2} = w_{3,3}$ .  $\square$

**Lemma 5.12.**

$$w_{2,2} = w_{2,3}, \quad 2w_{2,2} = 2w_{2,3} = 0.$$

*Proof.* The equation  $w_{2,2} = w_{2,3}$  follows from Lemma 5.11 and Equation (5.3).

Next, we prove  $2w_{2,2} = 2w_{2,3} = 0$ . Since  $s_1$  and  $r_{-1,2}$  commute by the relation (P10)(c), it must be  $(1 - s_1)d(r_{-1,2}) = (1 - r_{-1,2})d(s_1)$ . Since

$$((1 - r_{-1,2})d(s_1))_2 = 2w_{2,2} \text{ while } (1 - s_1)d(r_{-1,2})_2 = 0,$$

we obtain  $2w_{2,2} = 2w_{2,3} = 0$ . □

**Lemma 5.13.**

$$4w_{1,1} = 2w_{2,4} = 2w_{3,1} = 2w_{4,5} = 0.$$

*Proof.* As in (P8),  $t_1$  and  $s_1 t_1 s_1$  commute. Thus it must be  $(1-t_1)d(s_1 t_1 s_1) = (1-s_1 t_1 s_1)d(t_1)$ . Since

$$\begin{aligned} d(s_1 t_1 s_1) &= d(s_1) + s_1 d(t_1) + s_1 t_1 d(s_1) \\ &= (w_{2,1} + w_{2,2} - w_{2,4} - w_{3,1})x_1 + (w_{2,1} + w_{2,2} + w_{3,2})x_2 + w_{3,3}x_3 + w_{2,4}(y_1 + y_2), \end{aligned}$$

we have

$$\begin{aligned} (1-t_1)d(s_1 t_1 s_1) &= -(w_{3,1} + w_{3,2})(x_1 - x_2) - 2w_{2,4}x_1, \text{ while} \\ (1-s_1 t_1 s_1)d(t_1) &= (w_{3,1} - w_{3,2})(x_1 - x_2). \end{aligned}$$

Hence we have  $2w_{2,4} = 2w_{3,1} = 0$ . Since  $w_{2,4} = w_{4,5} = -2w_{1,1}$ , we also have  $4w_{1,1} = 2w_{2,4} = 2w_{4,5} = 0$ . □

**Lemma 5.14.**

$$w_{4,2} = 2w_{4,3} = 0.$$

*Proof.* By the relation  $r_{1,2}^2 = s_2 d_{1,2} s_2 d_{1,2}^{-1}$  in (P9), we have  $d(r_{1,2}^2) = d(s_2 d_{1,2} s_2 d_{1,2}^{-1})$ . First, we have

$$d(r_{1,2}^2) = d(r_{1,2}) + r_{1,2}d(r_{1,2}) = (-w_{4,2} + w_{4,5})x_1 + 2w_{4,3}x_3 + 2w_{4,6}y_3.$$

Next, we compute  $d(s_2 d_{1,2} s_2 d_{1,2}^{-1})$ . Since  $d_{1,3} = t_2 d_{1,2} t_2^{-1}$  and  $d_{2,3} = t_1 d_{1,3} t_1^{-1}$ , we have

$$d(d_{1,3}) = w_{1,1}(x_1 + x_3) \text{ and } d(d_{2,3}) = w_{1,1}(x_2 + x_3).$$

Recall that  $k_i = a_i a_{i+1} t_i d_{i,i+1}^{-1}$  for  $i = 1, 2$ . Hence we have

$$\begin{aligned} d(k_i) &= d(a_i a_{i+1} t_i d_{i,i+1}^{-1}) \\ &= d(a_i) + a_i d(a_{i+1}) + a_i a_{i+1} d(t_i) - k_i d(d_{i,i+1}) \\ &= d(t_i). \end{aligned}$$

Since  $s_{i+1} = k_i s_i k_i^{-1}$  for  $i = 1, 2$ ,

$$d(s_{i+1}) = d(k_i) + k_i d(s_i) - s_{i+1} d(k_i) = k_i d(s_i).$$

Thus we obtain

$$\begin{aligned} d(s_2 d_{1,2} s_2 d_{1,2}^{-1}) &= (1 + s_2 d_{1,2})d(s_2) + s_2(1 - d_{1,2} s_2 d_{1,2}^{-1})d(d_{1,2}) \\ &= -2w_{1,1}x_2 + w_{2,4}(x_1 + x_2) \\ &= w_{2,4}x_1. \end{aligned}$$

Comparing  $d(r_{1,2}^2)$  and  $d(s_2 d_{1,2} s_2 d_{1,2}^{-1})$ , we have  $w_{4,2} = 2w_{4,3} = 0$ . □

**Lemma 5.15.**

$$w_{3,1} + w_{3,2} = w_{4,5}, w_{3,3} = w_{4,3} \text{ and } w_{4,6} = 0.$$

In particular,

$$2w_{3,1} = 2w_{3,2} = 2w_{3,3} = 2w_{4,3} = 0.$$

*Proof.* The relation  $t_1 r_{1,2} t_1 = r_{1,2} t_1 r_{1,2}$  in (P11) shows

$$(1 - t_1 + r_{1,2} t_1) d(r_{1,2}) = (1 - r_{1,2} + t_1 r_{1,2}) d(t_1).$$

A straightforward calculation shows

$$(1 - r_{1,2} + t_1 r_{1,2}) d(t_1) = (w_{3,1} + w_{3,2}) x_2 + w_{3,3} x_3, \text{ and}$$

$$(1 - t_1 + r_{1,2} t_1) d(r_{1,2}) = w_{4,5} x_2 + w_{4,3} x_3 + w_{4,6} y_3.$$

Thus we have  $w_{3,1} + w_{3,2} = w_{4,5}$ ,  $w_{3,3} = w_{4,3}$ , and  $w_{4,6} = 0$ . □

**Lemma 5.16.**

$$w_{2,1} = 0.$$

*Proof.* As in (P12),  $r_{1,3} = s_3 c_{1,3} s_3 c_{1,3}^{-1} k_2 a_3 a_1 t_2 d_{1,2}^{-1} t_2^{-1} r_{1,2}^{-1} s_2 h_2 r_{1,2}^{-1} h_2^{-1} k_2^{-1}$ , where  $h_2 = k_2^{-1} t_1^{-1} k_2$ . Since

$$(5.4) \quad \begin{aligned} d(h_2) &= w_{3,1} x_1 + w_{3,2} x_2 + w_{3,3} x_3 \text{ and} \\ d(s_3 c_{1,3} s_3 c_{1,3}^{-1}) &= w_{2,4} (x_1 + x_2), \end{aligned}$$

we have

$$\begin{aligned} d(r_{1,3}) &= (-w_{2,1} + w_{2,2}) x_1 + (w_{2,3} + w_{3,2}) x_2 - w_{2,1} x_3 + w_{4,5} y_3 \text{ and} \\ d(r_{1,3}^2) &= d(r_{1,3}) + r_{1,3} d(r_{1,3}) = (-w_{2,1} + w_{4,5}) x_1 + (w_{2,1} + w_{4,5}) x_2 \end{aligned}$$

by a straightforward calculation. The relation  $r_{1,3}^2 = s_3 c_{1,3} s_3 c_{1,2}^{-1}$  in (P9) and the equations (5.4) show that  $w_{2,1} = 0$ . □

**Lemma 5.17.**

$$w_{3,2} = w_{3,3} = w_{4,3} = 0.$$

In particular,

$$w_{2,4} = w_{3,1} = w_{4,5} = 2w_{1,1}.$$

*Proof.* It is sufficient to prove that  $w_{3,2} = 0$ . Recall that  $z = (a_1 a_2 a_3) s_1 t_1 t_2 s_1 t_1 s_1 c_{1,3}$  and  $z_3 = k_2 k_1 z$ . Hence we have

$$d(z) = w_{2,4} (x_1 + x_2 + x_3 + y_1 + y_2 + y_3) + w_{3,3} x_1 + w_{3,1} x_2 + w_{3,2} x_2,$$

and  $d(z_3) = d(z)$ . Since  $r_{1,3}$  and  $z_3$  commute by the relation (P10)(e), it must be  $(1 - r_{1,3}) d(z_3) = (1 - z_3) d(r_{1,3})$ . A straightforward calculation shows

$$(1 - r_{1,3}) d(z_3) = w_{3,2} (x_1 + x_2)$$

while  $(1 - z_3) d(r_{1,3}) = 0$ . Thus we obtain  $w_{3,2} = 0$ . □

Summarizing Lemmas 5.10, 5.11, 5.12, 5.13, 5.14, 5.15, 5.16 and 5.17, we have

$$\begin{aligned} d(a_1) &= w_{1,1}x_1, d(s_1) = w_{2,2}(x_2 + x_3) + 2w_{1,1}y_1, \\ d(t_1) &= 2w_{1,1}x_1, \text{ and } d(r_{1,2}) = 2w_{1,1}y_2 \end{aligned}$$

where  $4w_{1,1} = 2w_{2,2} = 0$ . It can be verified that such  $d$  is compatible with the relations (P1)–(P12). Now we have

$$H^1(\mathcal{H}_3; H_A) \cong \text{Ker } f_3 \cong \{(w_{1,1}, w_{2,2}) \in A^2; 4w_{1,1} = 2w_{2,2} = 0\}.$$

**Proposition 5.18.**

$$H_1(\mathcal{H}_3; H/L) \cong (\mathbb{Z}/2\mathbb{Z})^2,$$

and the image of the homomorphism  $H_2(\mathcal{H}_3; H/L) \rightarrow H_1(\mathcal{H}_3; L)$  induced by the exact sequence  $0 \rightarrow L \rightarrow H \rightarrow H/L \rightarrow 0$  is isomorphic to  $\mathbb{Z}/2\mathbb{Z}$ .

*Proof.* As we saw in the proof of Theorem 1.1, under the isomorphism

$$H^1(\mathcal{H}_3; H_A) \cong \{(w_{1,1}, w_{2,2}) \in A^2; 4w_{1,1} = 2w_{2,2} = 0\},$$

the submodule  $\{(w_{1,1}, w_{2,2}) \in A^2; 2w_{1,1} = 2w_{2,2} = 0\}$  is in  $\text{Im}(H^1(\mathcal{H}_3; L_A) \rightarrow H^1(\mathcal{H}_3; H_A))$ . The universal coefficient theorem implies  $\text{Im}(H_1(\mathcal{H}_3; H) \rightarrow H_1(\mathcal{H}_3; H/L))$  is of order at least 4. On the other hand,  $H_1(\mathcal{H}_3; H/L)$  is at most order 4 as explained in Remark 3.9. Thus we obtain  $H_1(\mathcal{H}_3; H/L) \cong (\mathbb{Z}/2\mathbb{Z})^2$ .

By Lemma 2.2, the coinvariant  $H_0(\mathcal{H}_3; L)$  is trivial. Thus the homomorphism  $H_1(\mathcal{H}_3; H) \rightarrow H_1(\mathcal{H}_3; H/L)$  is surjective. Since  $H_1(\mathcal{H}_3; H) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ , we have  $H_1(\mathcal{H}_3; L) \cong (\mathbb{Z}/2\mathbb{Z})^2$  and  $H_1(\mathcal{H}_3; H/L) \cong (\mathbb{Z}/2\mathbb{Z})^2$ . The exact sequence

$$H_2(\mathcal{H}_3; H/L) \longrightarrow H_1(\mathcal{H}_3; L) \longrightarrow H_1(\mathcal{H}_3; H) \longrightarrow H_1(\mathcal{H}_3; H/L) \longrightarrow 0$$

shows  $\text{Im}(H_2(\mathcal{H}_3; H/L) \rightarrow H_1(\mathcal{H}_3; L)) \cong \mathbb{Z}/2\mathbb{Z}$ .  $\square$

## 6. PROOF OF THEOREM 1.2

In this section, we prove Theorem 1.2, and calculate  $H_1(\mathcal{H}_g^*; L)$  and  $H_1(\mathcal{H}_g^*; H/L)$ .

**Lemma 6.1.**

$$(L \otimes L^*)_{\mathcal{H}_g} \cong (L \otimes H)_{\mathcal{H}_g} \cong \mathbb{Z}.$$

*Proof.* The action of  $\mathcal{H}_g$  on  $L$  factors through  $\text{GL}(g; \mathbb{Z})$ , and  $L$  is isomorphic to  $V = \mathbb{Z}^g$  endowed with the natural left  $\text{GL}(g; \mathbb{Z})$ -module. Thus, the fact that  $(V \otimes V^*)_{\text{GL}(g; \mathbb{Z})} \cong \mathbb{Z}$  implies  $(L \otimes L^*)_{\mathcal{H}_g} = \mathbb{Z}$ .

Next, recall that the intersection form on  $H$  induces an isomorphism  $L^* \cong H/L$ . Since there is an exact sequence

$$(L^{\otimes 2})_{\mathcal{H}_g} \longrightarrow (L \otimes H)_{\mathcal{H}_g} \longrightarrow (L \otimes L^*)_{\mathcal{H}_g} \longrightarrow 0,$$



it suffices to show that  $\text{Im}((L^{\otimes 2})_{\mathcal{H}_g} \rightarrow (L \otimes H)_{\mathcal{H}_g})$  is trivial. Since this image is generated by  $x_i \otimes x_j$  for  $1 \leq i \leq g$  and  $1 \leq j \leq g$ , and

$$a_j(x_i \otimes y_j) - x_i \otimes y_j = x_i \otimes x_j,$$

we obtain  $\text{Im}((L^{\otimes 2})_{\mathcal{H}_g} \rightarrow (L \otimes H)_{\mathcal{H}_g}) = 0$ .  $\square$

*Proof of Theorem 1.2.* The exact sequence  $1 \rightarrow \mathbb{Z} \rightarrow \mathcal{H}_{g,1} \rightarrow \mathcal{H}_g^* \rightarrow 1$  induces an exact sequence

$$H_1(\mathbb{Z}; H)_{\mathcal{H}_g^*} \longrightarrow H_1(\mathcal{H}_{g,1}; H) \longrightarrow H_1(\mathcal{H}_g^*; H) \longrightarrow 0.$$

Since  $H_1(\mathbb{Z}; H)_{\mathcal{H}_g^*} \cong H_{\mathcal{H}_g^*} = 0$ , we obtain an isomorphism  $H_1(\mathcal{H}_{g,1}; H) \cong H_1(\mathcal{H}_g^*; H)$ .

Next, we compute  $H_1(\mathcal{H}_g^*; H)$ . Morita showed that  $H_1(\mathcal{M}_g^*; H) \cong \mathbb{Z}$  and the forgetful exact sequence  $1 \rightarrow \pi_1 \Sigma_g \rightarrow \mathcal{M}_g^* \rightarrow \mathcal{M}_g \rightarrow 1$  induces an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow H_1(\mathcal{M}_g^*; H) \longrightarrow H_1(\mathcal{M}_g; H) \longrightarrow 0$$

when  $g \geq 2$ . In Lemma 6.1, we showed the isomorphism  $H_1(\pi_1 \Sigma_g; H)_{\mathcal{H}_g^*} \cong \mathbb{Z}$  induced by the intersection form on  $H$ . Thus, restricting the exact sequence to  $\mathcal{H}_g^*$ , we obtain a commutative diagram

$$\begin{array}{ccccccc} \mathbb{Z} & \longrightarrow & H_1(\mathcal{H}_g^*; H) & \longrightarrow & H_1(\mathcal{H}_g; H) & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & H_1(\mathcal{M}_g^*; H) & \longrightarrow & H_1(\mathcal{M}_g; H) \longrightarrow 0. \end{array}$$

By the above diagram, both the kernels and the cokernels of the homomorphisms  $H_1(\mathcal{H}_g^*; H) \rightarrow H_1(\mathcal{M}_g^*; H)$  and  $H_1(\mathcal{H}_g; H) \rightarrow H_1(\mathcal{M}_g; H)$  coincide. By Remark 3.2 and Theorem 1.1, we see that  $\text{Coker}(H_1(\mathcal{H}_g^*; H) \rightarrow H_1(\mathcal{M}_g^*; H))$  is trivial for  $g \geq 2$  and that  $\text{Ker}(H_1(\mathcal{H}_g^*; H) \rightarrow H_1(\mathcal{M}_g^*; H))$  is trivial when  $g \geq 4$  and is isomorphic to  $\mathbb{Z}/2\mathbb{Z}$  when  $g = 2, 3$ . Thus we can determine  $H_1(\mathcal{H}_g^*; H)$ .  $\square$

**Proposition 6.2.** (1) When  $g \geq 2$ , the forgetful homomorphism  $\mathcal{H}_g^* \rightarrow \mathcal{H}_g$  induces an isomorphism

$$H_1(\mathcal{H}_g^*; H/L) \cong H_1(\mathcal{H}_g; H/L).$$

In particular, we have

$$H_1(\mathcal{H}_g^*; H/L) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } g \geq 4, \\ (\mathbb{Z}/2\mathbb{Z})^2 & \text{if } g = 2, 3. \end{cases}$$

(2) When  $g \geq 4$ , the homomorphism  $H_1(\mathcal{H}_g^*; L) \rightarrow H_1(\mathcal{H}_g^*; H)$  induced by the inclusion  $L \rightarrow H$  is injective. When  $g = 2, 3$ ,  $\text{Ker}(H_1(\mathcal{H}_g^*; L) \rightarrow H_1(\mathcal{H}_g^*; H)) \cong \mathbb{Z}/2\mathbb{Z}$ . In particular, we have

$$H_1(\mathcal{H}_g^*; L) \cong \begin{cases} \mathbb{Z} & \text{if } g \geq 4, \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } g = 2, 3. \end{cases}$$

*Proof.* Consider the exact sequences between homology groups with coefficients in  $L$  induced by the forgetful exact sequences  $1 \rightarrow \pi_1 \Sigma_g \rightarrow \mathcal{M}_g^* \rightarrow \mathcal{M}_g \rightarrow 1$  and its restriction  $1 \rightarrow \pi_1 \Sigma_g \rightarrow \mathcal{H}_g^* \rightarrow \mathcal{H}_g \rightarrow 1$ . Applying Lemma 6.1, we obtain a commutative diagram

$$\begin{array}{ccccccc} \mathbb{Z} & \longrightarrow & H_1(\mathcal{H}_g^*; L) & \longrightarrow & H_1(\mathcal{H}_g; L) & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & H_1(\mathcal{H}_g^*; H) & \longrightarrow & H_1(\mathcal{H}_g; H) \longrightarrow 0. \end{array}$$

By the above diagram, both of the kernels and the cokernels of the homomorphisms  $H_1(\mathcal{H}_g^*; L) \rightarrow H_1(\mathcal{H}_g^*; H)$  and  $H_1(\mathcal{H}_g; L) \rightarrow H_1(\mathcal{H}_g; H)$  coincide. In particular, by Lemma 2.2, we obtain

$$\begin{aligned} H_1(\mathcal{H}_g^*; H/L) &\cong \text{Coker}(H_1(\mathcal{H}_g^*; L) \rightarrow H_1(\mathcal{H}_g^*; H)) \\ &\cong \text{Coker}(H_1(\mathcal{H}_g; L) \rightarrow H_1(\mathcal{H}_g; H)) \\ &\cong H_1(\mathcal{H}_g; H/L). \end{aligned}$$

In Remark 3.10 and Propositions 5.9 and 5.18, We see that  $\text{Ker}(H_1(\mathcal{H}_g; L) \rightarrow H_1(\mathcal{H}_g; H))$  is trivial when  $g \geq 4$ , and is isomorphic to  $\mathbb{Z}/2\mathbb{Z}$ . In Lemma 3.7 and Propositions 5.9 and 5.18, we also see that  $\text{Coker}(H_1(\mathcal{H}_g; L) \rightarrow H_1(\mathcal{H}_g; H))$  is isomorphic to  $\mathbb{Z}/2\mathbb{Z}$  when  $g \geq 4$ , and is isomorphic to  $(\mathbb{Z}/2\mathbb{Z})^2$  when  $g = 2, 3$ . Thus, we can determine  $H_1(\mathcal{H}_g^*; L)$ .  $\square$

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